MATH 817 Notes

JD Nir
jnir@huskers.unl.edu
www.math.unl.edu/∼jnir2/817.html
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Let B be a basis of an F vector space V, and let W be any F-vector space. Then the function

$$\mathrm{Hom}_F(V,W) \to \mathrm{Functions}(B,W)$$

given by $\varphi \mapsto \varphi|_B$ is bijective.

Say $B = (v_1, v_2, \ldots, v_n)$ is a basis of V and $C = (w_1, w_2, \ldots, w_m)$ is a basis of W.

$$\begin{array}{ccccc} \mathrm{Hom}_F(V,W) & \overset{\text{1-1}}{\underset{\text{onto}}{\longleftrightarrow}} & \mathrm{Functions}(B,W) & \overset{\text{1-1}}{\underset{\text{onto}}{\longleftrightarrow}} & \mathrm{Mat}_{m\times n}(F) \\ \varphi & \mapsto & \varphi|_B & & \\ & & (g : B \to W) & \mapsto & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \vdots & & \vdots \\ \vdots & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

Notation

- Given V, W and B, C as above, for $\varphi \in \mathrm{Hom}_F(V,W)$ $M_B^C(\varphi) := [a_{ij}]_{m\times n}, \varphi(v_i) = a_{1i}w_1 + a_{2i}w_2 + \cdots + a_{mi}w_m$.
- For $v \in V$, $[v]_B = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$ where $v = \lambda_1 b_1 + \cdots + \lambda_n b_n$

Prop Let V, W, B, C be as above.

1 $V \underset{\substack{\uparrow \\ \text{isomorphism of vector space}}}{\cong} F^n$ via $v \mapsto [v]_B$

$$\sum \lambda_i v_i \leftarrow\!\shortmid \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

2 There is a bijection $\mathrm{Hom}_F(V,W) \cong \mathrm{Mat}_{m\times n}(F)$ given by $\varphi \mapsto M_B^C(\varphi)$

3 $\forall v \in V, \forall \varphi \in \mathrm{Hom}_F(V,W)$

$$[\varphi(v)]_C = M_B^C(\varphi) \cdot [v]_B$$

4 If $A = (u_1, u_1, \ldots, u_p)$ is a basis of another F-vector space U, then $\forall \varphi \in \mathrm{Hom}_F(V,W)$, $\forall \psi : \mathrm{Hom}_F(U,V)$,

$$\mathrm{Mat}_A^C(\varphi \circ \varphi) = \mathrm{Mat}_B^C(\varphi) \underset{\substack{\uparrow \\ \text{matrix multiplication}}}{\cdot} \mathrm{Mat}_A^B(\psi)$$

Note A, B, C are ordered bases in this proposition.

Cor 1: Matrix multiplication is associative.

Cor 2: $\mathrm{Hom}_F(V,V) \cong \mathrm{Mat}_{n\times n}(F)$ and also $\mathrm{Aut}_F(V) \underset{\substack{\uparrow \\ \text{isomorphism of groups.}}}{\cong} GL_n(F)$

Similarity: Let B, B' are two ordered bases of V ($\#B = \#B' < \infty$). Let $P = M_B^{B'}(\mathrm{id}_v)$, $\mathrm{id}_V : V \to V$ is identity map.

Then

- P is invertible and $P^{-1} = M_{B'}^{B}(\mathrm{id}_V)$. Why? $M_{B'}^{B}(\mathrm{id}_v) \cdot M_B^{B'}(\mathrm{id}_v) \overset{\text{Prop}}{=} M_B^B(\mathrm{id}_v) = I_n$ and $M_B^{B'}(\mathrm{id}_v) \cdot M_{B'}^{B}(\mathrm{id}_v) = I_n$
- For any $\varphi \in \mathrm{Hom}_F(V,V)$,

$$\begin{aligned} P \cdot M_B^B(\varphi) \cdot P^{-1} &= M_B^{B'}(\mathrm{id}_V) \cdot M_B^B(\varphi) \cdot M_{B'}^{B}(\mathrm{id}_V) \\ &= M_B^{B'}(\mathrm{id}_V \cdot \varphi) \cdot M_{B'}^{B}(\mathrm{id}_V) \\ &= M_{B'}^{B'}(\mathrm{id}_V \circ \varphi \circ \mathrm{id}_V) \\ &= M_{B'}^{B'}(\varphi) \end{aligned}$$

So, similarity of matricies amounts to a change of basis:

Given base B, B' of V, if $\varphi \in \mathrm{Hom}_F(U,V)$, then

$$M_B^B(\varphi) \underset{\substack{\uparrow \\ \text{similarity}}}{\sim} M_{B'}^{B'}(\varphi)$$

Equivalence Let V, W be two vector spaces, say B, B' are two ordered bases of V and C, C' are two ordered bases of W.

$\forall \varphi \in Q \cdot \mathrm{Hom}_F(V,W), M_B^C(\varphi) \cdot P = M_{B'}^{C'}(\varphi)$

where $\begin{array}{rcl} P &=& M_{B'}^{B}(\mathrm{id}_V) \\ Q &=& M_C^{C'}(\mathrm{id}_w) \end{array}$.

If A, A' are two $m \times n$ matricies, then A and A' are equivalent if

$$\underset{m\times n}{A'} = Q \underset{m\times n}{A} P$$

for some $Q \in GL_n(F), P \in GL_n(F)$.

Note: $\forall Q \in GL_n(F)$, QA is obatained from A via element row operations.

$\forall P \in GL_n(F)$, AP is obtained from A via elementary column operations.

Def Say W is a subspace of a F-vector space V then $V/W = \frac{(V,+)}{(W,+)}$ is a vector space via $\lambda \cdot (v + W) := \lambda v + W$.