MATH 817 Notes

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$\mathbb{R}$ is a $\mathbb{Q}$-vector space and so it has a basis:

$\exists$ a subset B of $\mathbb{R}$ such that every real number is unqiuely expressible as a $\mathbb{Q}$-linear combination of some finite subset B.

B cannot be described. B is uncountable.

Def A vector space V over a field F is finite dimensional if $\exists$ a finite subset S such that $\text{Span}(S) = V$.

Note By Theorem$'$, V has a finite basis.

Theorem If V is a finite dimensional vector space , then any two basis of V have the same (finite) cardinality.

Pf: Let $A = (a_1, \dots, a_n)$ be any finite basis of V and let B be any basis of V.

Step 1: For any $b_1 \in B$, after reordering A, $A' = (b_1, a_2, a_3, \dots, a_n)$ is also a basis.

* $b_1 = \sum_{i=1}^{n} c_i a_i$. $b_1 \neq 0 \to \exists i, c_i \neq 0$. Reorder so that $c_1 \neq 0$. Then $a_1 = b_1 - \frac{c_2}{c_1} a_2 - \cdots - \frac{c_n}{c_1} a_n \Rightarrow$ $a_1 \in \text{Span}(A') \Rightarrow \text{Span}(A') = V$.
  If $\lambda_1 b_1 + \lambda_2 a_2 + \lambda_3 a_3 + \cdots + \lambda_n a_n = 0$, then $\lambda_1 c_1 a_1 + (\lambda_1 c_2 + \lambda_2) a_2 + \cdots = 0$. $\therefore$ $\lambda_1 c_1 = 0$. $\therefore$ $\lambda_1 = 0$ $\therefore$ $\lambda_2 = \cdots = \lambda_n = 0$. ✓

Step 2: For any $b_2 \neq b_1, b_2 \in B$, after reordering $a_2, \dots, a_n$, $A'' = (b_1, b_2, a_3, a_4, a_5, \dots, a_n)$ is also a basis.

* $b_2 = c_1 b_1 + c_2 a_2 + c_3 a_3 + \cdots + c_n a_n$. If $c_2 = c_3 = \cdots = c_n = 0$, I reach a contradiction to B being linearly independent. Reorder so $c_2 \neq 0$.
  I claim $A'' = (b_1, b_2, a_3, a_4, \dots, a_n)$ is a basis:
  - $a_2 \in \text{Span}(A'') \Rightarrow \text{Span}(A'') \supseteq \text{Span}(A') = V$
  - If $\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 a_3 + \cdots + \lambda_n a_n = 0$ then $(\lambda_1 + \lambda_2 c_1)\underline{b_1} + \lambda_2 c_2 \underline{a_2} + (\lambda_3 + \lambda_2 c_3)\underline{a_3} + \cdots = 0$ $\therefore$ $\lambda_2 = 0$ and so $\lambda_i = 0$ $\forall i$.

Step 3: $\forall b_3 \notin \{b_1, b_2\}$, upon reordering $a_3, \dots, a_n$ $A''' = (b_1, b_2, b_3, a_4, a_5, \dots, a_n)$ is a basis too.

The proof of step 3 is nearly identitcal to that of step 2.

This process can be continued, provided $\#B \geq n$, n times to get:

$\forall b_1, \dots, b_n \in B$, distinct elements, $(b_1, \dots, b_n)$ is a basis of V.

So $B = \{b_1, \dots, b_n\}$.

If $\#B < n$, redo whole argument with roles of A and B switched. □

Def If V and W are F-vector spaces, a linear transformation from V to W is a function $\varphi : V \to W$ such that

- $\varphi(v + v') = \varphi(v) + \varphi(v')$ $\forall v, v' \in V$

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- $\varphi(c \cdot v) = c \cdot \varphi(v)\ \forall v \in V_1\ c \in F$

Ex (1) $\int_0^1 (—)\, dx : \mathscr{C}[0,1] \to \mathbb{R}$ is linear.

(2) $F =$ any field. $V = F^n\ W = F^m$

$A =$ any $m \times n$ matrix in F $T_A : V \to W$ defined by $T_A(v) = A \cdot v$ is F-linear.

Theorem [UMP of a basis]

If V is a F-vector space, B is a basis of V and W is any other F-vector space, then the function

$$\mathrm{Hom}_F(V,W) \to \mathrm{Functions}(B,W),$$

given by $\varphi \mapsto (\varphi|_B : B \to W)$, is bijective.

Note $\mathrm{Hom}_F(V,W) := \{\varphi : V \to W \mid U \text{ is a } F\text{-linear transformation.}\}$

$\mathrm{Functions}(A,B) = \{g : A \to B\}$

So, given any function $g : B \to W$ $\exists!$ an F-linear transformation $\varphi_g : V \to W$ so that $\varphi_g(b) = g(b)\ \forall b \in B$.

Pf: If $g : B \to W$ is any function, define $\varphi_g : V \to W$ by

$$\varphi_g(v) = \sum_{b \in B} c_b g(b)$$

where $v = \sum_{b \in B} c_b \cdot b$, $c_b \in F$, is the unique expression of this form.

One checks that $\varphi_g \in \mathrm{Hom}_F(V,W)$.

Let $\mathrm{Hom}_F(V,W) \underset{\psi'}{\overset{\psi}{\rightleftarrows}} \mathrm{Functions}(B,W)$ be defined as $\psi(\varphi) = \varphi|_B$ and $\psi'(g) = \varphi_g$.

$\psi(\psi'(g)) = \varphi_g|_B = g$ since $\varphi_g(b) = g(b)$. ✓

$\psi'(\psi(\varphi)) = \varphi_{(\varphi|_B)} = \varphi$, since $\varphi_{(\varphi|_B)}$ and φ agree on B. □