MATH 817 Notes

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Problem Set #8:

- Due Monday
- Final version posted soon

Def F = a field. An F-vector space is an abelian group $(V, +)$ with a function $F \times V \to V$, written $(c, v) \mapsto c \cdot v$ and called "scalar multiplication", such that $\forall c, d \in F, v, w \in V$

(1) $c \cdot (v + w) = c \cdot v + c \cdot w$

(2) $(c \cdot d) \cdot v = c \cdot (d \cdot v)$

(3) $1 \cdot v = v$

(4) $(c + d) \cdot v = c \cdot v + d \cdot v$

Note $0_F \cdot v = 0_V$

$0_F \cdot v = (0_F + 0_F) \cdot v = 0_F \cdot v + 0_F \cdot v$

Note (1) - (3) $\underbrace{(F \setminus \{0\}, \cdot)}_{\text{Group}}$ acts on $(V, +)$ via automorphisms

Ex (1) $F^n = \left\{ \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \mid c_i \in F \right\}$, with obvious values for $+$ and $\cdots$.

$$\lambda \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \lambda c_1 \\ \vdots \\ \lambda c_n \end{pmatrix}$$

(2) $\mathbb{C}$ is a vector space "over" $\mathbb{R}$

$\mathbb{C}$ is a vector space "over" $\mathbb{Q}$

$\mathbb{R}$ is a vector space "over" $\mathbb{Q}$

(3) $F = \mathbb{Z}/p, p$ prime. An F-vector space is the same thing as an e.a.p.g. If $(V, +)$ is an e.a.p.g., define scaling by:

$$\overline{n} \cdot v := \underbrace{v + \cdots + v}_{n}$$

(4) $\mathcal{C}[0, 1] = \{f : [0, 1] \to \mathbb{R} \mid f \text{ is continuous}\}$ is a $\mathbb{R}$-vector space.

Def $V = F$-vector space.

A subspace of V is a subset W that is closed under +,, closed under scaling, and $0 \in W$.

Note $(-1) \cdot w = -w \ \forall w \in V$

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Any possibly infinite intersection of subspaces of V is again a subspace.

Def If S is a subset of V, $\text{span}(S) = \bigcap_{W \text{ is a subspace of } V} W$

$\text{span}(S) = \{c_1 s_1 + \cdots + c_n s_n \mid n \geq 0, c_i \in F, s_i \in S\}$

The empty span $= 0$

$\text{span}(\varnothing) = \{0\}$

Def A subset S of V is linearly independent if whenever $c_1 s_1 + \cdots + c_n s_n = 0$ for some n and some $c_i \in F, s_i \in S$, we have $c_1 = c_2 = \cdots = c_n = 0$.

A subset B of V is a basis if

- $\text{span}(B) = v$
- B is linearly independent

ex F^n

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \text{ - } i$$

$\{e_1, e_2, \ldots, e_n\}$ is a basis of F^n

Prop If B is a basis of V then every $v \in V$ is uniquely expressaible as

$$v = \sum_{b \in B} c_b \cdot b, c_b \in F,$$

where $c_b = 0$ for all but a finite number of b's.

Pf Existence holds since $\text{span}(B) = V$.

If $\sum_{b \in B} c_b \cdot b = \sum_{b \in B} \lambda_b \cdot b$, then

$$\sum_{b \in B} (c_b - \lambda_b) \cdot b = 0.$$

Since B is linearly independent, $c_b = \lambda_b, \forall b$

underlineTheorem Every vector space has a basis.

Use Zorn's Lemma: If P is a poset such that every totally ordered subset T of P has an upper bound in P (i.e., $\exists p \in P$ such that $t \leq p \ \forall t \in T$), P has at least one maximal element.

T totally ordered: If $t, t' \in T$, then $t \leq t'$ or $t' \leq t$.

Pf of Theorem: Let $P = \{S \mid S \overset{\text{subset}}{\subseteq} V, S \text{ is linearly independent}\}$. P is a psoet under $\subseteq$. Let T be a totally ordered subset of P. Let $X = \bigcup_{y \in T} Y$. I claim $X \in P$. Say $c_1 x_1 + \cdots + c_n x_n = 0$, $c_i \in F, x_i \in X$. $x_i \in Y_i$, since $Y_i \in T, \forall i$. Since T is totally ordered, $\exists i$ so that $Y_i \supset Y_j \ \forall j = 1, \leq, n$.

$\therefore x_1, \ldots, x_n \in Y_i$. Y_i is linearly independent. $\therefore\ c_1 = \cdots = c_n = 0$.

By Zorn's Lemma, P has a maximal element B.

Claim span$(B) = V$. Say not. Let $v \in V \setminus \text{span}(B)$. $B' = B \cup \{v\}$. I now claim B' is linearly independent. Say $c \cdot v + c_1 b_1 + \cdots + c_n b_n = 0$, $c, c_i \in F, b_i \in B$. If $c \neq 0$, $v = -\frac{c_1}{c} b_1 - \cdots - \frac{c_n}{c} b_n \in \text{span}\, B$.

So, $c = 0$ and thus $c_1 = \cdots = c_n = 0$. □

Theorem′: If $L \subset S$ are subsets of a vector space V such that L is linearly independent and $\text{span}(S) = V$, then $\exists$ a basis B of V such that $L \subseteq B \subseteq S$.