MATH 817 Notes

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Exam I: Tuesday, 2 hours

Do we want $\frac{1}{2}$ the exam to be taken from a pre-released list of ~ 15 old qual. problems?

Thm $\#G = p^r, r \geq 1 \Rightarrow Z(G) \neq \{e\}$ "non-trivial p groups have non-trivial centers"

Cor: If a group G has order p^2, p a prime, then G is abelian.

Note In fact, $\#G = p^2$, then $G \cong \mathbb{Z}/p^2$ or $G \cong \mathbb{Z}/p \times \mathbb{Z}/p$.

Proof of Cor: By Theorem + Lagrange, $\#Z(G) = p$ or p^2. If $\#Z(G) = p$, then $\#\left(G/Z(G)\right) = p$ and hence $G/Z(G)$ is cyclic. So, the result follows from the Lemma. □

Lemma: If G is any group and $G/Z(G)$ is cyclic, then G is abelian.

Pf: Say $G/Z(G) = \langle g \cdot Z(G) \rangle$. If $x, y \in G$, then $xZ(G) \begin{array}{l} = (g \cdot Z(G))^i \\ = g^i \cdot Z(G) \end{array}$, some i.

$\therefore\ x = g^i \cdot z$, some $i \in \mathbb{Z}$, $z \in Z(G)$

Likewise, $y = g^j \cdot z'$, $j \in \mathbb{Z}, z' \in Z(G)$

$xy = g^i z g^j z' = g^{i+j} z z' = yx$ □

Conjugacy classes in A_5.

Recall: For S_5, the conjugacy classes are

$$[e]_{s_5}, [(1\ 2)]_{S_5}, [(1\ 2\ 3)]_{S_5}, [(1\ 2\ 3\ 4)]_{S_5}, [(1\ 2\ 3\ 4\ 5)]_{S_5}, [(1\ 2)(3\ 4)]_{S_5}, [(1\ 2)(3\ 4\ 5)]_{S_5}$$

of lengths $1, 10, 20, 20, 30, 24, 15, 20$

Note If $\sigma \in A_5, [\sigma]_{A_5} \subseteq [\sigma]_{S_5}$

where $\begin{array}{lcl} [\sigma]_{A_5} & = & \left\{\tau\sigma\tau^{-1} \mid \tau \in A_5\right\} \\ [\sigma]_{S_5} & = & \left\{\tau\sigma\tau^{-1} \mid \tau \in S_5\right\} \end{array}$

Lemma The conjugacy classes $\begin{array}{llllll} \text{of } A_5 \text{ are} & [e]_{A_5}, & [(1\ 2\ 3)]_{A_5}, & [(1\ 2)(3\ 4)]_{A_5}, & [(1\ 2\ 3\ 4\ 5)]_{A_5} & [(2\ 1\ 3\ 4\ 5)]_{A_5} \\ \text{of lengths} & 1, & 20, & 15, & 12, & 12. \end{array}$

Proof: I claim $[(1\ 2\ 3)]_{A_5} = [(1\ 2\ 3)]_{S_5}$ and $[(1\ 2)(3\ 4)]_{A_5} = [(1\ 2)(3\ 4)]_{S_5}$. Since $\subseteq$ holds, it suffices to show $\#[(1\ 2\ 3)]_{A_5} = 20$ and $\#[(1\ 2)(3\ 4)]_{A_5} = 15$.

$\#[(1\ 2\ 3)]_{S_5} = 20 \underset{\text{LOIS}}{\Rightarrow} [S_5 : C_{S_5}((1\ 2\ 3))] = 20 \Rightarrow \#C_{S_5}((1\ 2\ 3)) = 6$

Clearly, $(4\ 5), (1\ 2\ 3) \in C_{S_5}((1\ 2\ 3))$. So, $C_{S_5}((1\ 2\ 3)) \supseteq \langle (4\ 5), (1\ 2\ 3)\rangle = \{e, (4\ 5), (4\ 5)(1\ 2\ 3), (1\ 2\ 3), (1\ 3\ 2), (4\ 5)$
T

$\therefore C_{S_5}((1\ 2\ 3)) = T$

Observe $C_{A_5}((1\ 2\ 3)) - C_{S_5}((1\ 2\ 3)) \cap A_5 = \{e, (1\ 2\ 3), (1\ 3\ 2)\} = \langle (1\ 2\ 3)\rangle$

By LOIS, $\#[(1\ 2\ 3)]_{A_5} = \frac{60}{3} = 20$ ✓

Similarly, $\#C_{S_5}((1\ 2)(3\ 4)) = \frac{120}{15} = 8$ and we can show

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$$C_{S_5}((1\ 2)(3\ 4)) = \left\{ e, (1\ 2), (3\ 4), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3), \underset{\substack{\| \\ (1\ 3\ 2\ 4)}}{(1\ 2)(1\ 3)(2\ 4)}, \underset{\substack{\| \\ (1\ 4\ 2\ 3)}}{(3\ 4)(1\ 4)(2\ 3)} \right\}$$

$\therefore C_{A_5}((1\ 2)(3\ 4)) = 4 \overset{\text{LOIS}}{\Longrightarrow} [(1\ 2)(3\ 4)]_{A_5} = 15$ ✓

$\#C_{S_5}((1\ 2\ 3\ 4\ 5)) = 5 \Rightarrow C_{S_5}((1\ 2\ 3\ 4\ 5)) = \langle (1\ 2\ 3\ 4\ 5) \rangle$

$\Rightarrow \#C_{A_5}((1\ 2\ 3\ 4\ 5)) = 5 \overset{\text{LOIS}}{\Longrightarrow} \#[(1\ 2\ 3\ 4\ 5)]_{A_5} = \frac{60}{5} = 12$

Also, $[\sigma]_{A_5} = 12\ \forall$ five cycles σ.

Exercise: (1 2 3 4 5) and (2 1 3 4 5) are not conjugate in A_5. □

Theorem A_5 is simple.

Proof: Let $N \trianglelefteq A_5$. Then $gNg^{-1} = N\ \forall g$ and so N is a union of conjugacy classes.

$\therefore \#N = 1+$a sum of a sub-list of $(12, 12, 15, 20)$. Also, $\#N \mid 60$. $\#N = 1$ and $\#N = 60$ are the only possibilities. □