# MATH 817 Notes

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Exam Options

(a) Any two hour interval contains in [8:15am, 5:00pm] on Tuesday October 13

(b) Wed night, October 14, with proctor

(c) In class Wednesday and Friday, October 14th and October 16th, with a proctor on Friday.

$G = \{\text{rotational symmetries of a dodecahedron}\}$

$\exists$ 4 cubes inscribed in dodecahedron.

$G$ acts on $\{C_1, C_2, C_3, C_4, C_5\}$

$G_{C_1}$

$\#H = 12$

$\rho : G \to S_5$

Let $\sigma \in S_n$. $G := \langle \sigma \rangle \leq S_n$.

$S_n$ acts on $\{1, 2, \dots, n\} \Rightarrow G$ acts on $\{1, 2, \dots, n\}$

E.g. $\sigma = (2\ 3\ 4)(5\ 6) \in S_7$

$G = \langle \sigma \rangle$.

What are the orbits of the action of $G$ on $\{1, 2, 3, \dots, 7\}$?

$$\{1\}, \{2, 3, 4\}, \{5, 6\}, \{7\}$$

If $\sigma = \sigma_1 \circ \cdots \circ \sigma_m$ is the disjoint decomposition of $\sigma$ and $\sigma_i = (a_{i1}\ a_{i2}\ \cdots\ a_{i\ell_i}), \ell_i \geq 2\ \forall i$, then $(\ell_1, \ell_2, \dots, \ell_m)$ = list of orbits lengths of size at least 2 for the action of $\langle \sigma \rangle$ on $\{1, \dots, n\}$.

So, $(\ell_1, \ell_2, \dots, \ell_m)$ is uniquely determined by $\sigma$.

$G$ acts on itself via conjugation:

$X = G$

$g \bullet x := gxg^{-1}$ $g \in G, x \in X = G$.

- $g_1 \bullet (g_2 \bullet x) = g_1 \bullet (g_2 x g_2^{-1})$
  $= g_1 g_2 x g_2^{-1} g_1^{-1}$
  $= (g_1 g_2) \bullet x$ ✓
- $e \bullet x = x$ ✓

The stabilizer of $x$ is … $C_G(x) =$

$$\{g \mid gxg^{-1} = x\}$$

The orbits are conjugacy classes.

Prop For any $x \in G$, # of elements of $G$ that are conjugate to $x = [G : C_G(x)]$

Pf Apply LOIS.

Ex $S_{2n}$, $n$ odd

$r \in D_{2n}$

$srs^{-1} = srs = r^{-1}$

$\therefore r^{-1}$ is conjugate to $r$ ($r^{-1} \sim r$)

$C_G(f) \supset \langle r \rangle$ $[D_{2n} : \langle r \rangle] = 2$

$\Rightarrow [D_{2n} : C_G(r)] = \not{1}$ or 2

$[D_{2n} : C_G(x)] = 2$

$\{r, r^{-1}\}$ is a conjugacy class

So are $\{r^2, r^{-2}\}, \{r^3, r^{-3}\}, \ldots, \left\{r^{\frac{n-1}{2}}, r^{\frac{n+1}{2}}\right\}$

$r^{-3} = r^{n-3}$

$rsr^{-1} = sr^{-2} \therefore s \sim sr^{-2}$

$srs = r^{-1}$

$\Downarrow$

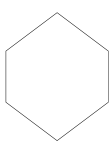
$rs = sr^{-1}$

$$\begin{aligned} r(sr^{-2})r^{-1} &= rsr^{-1} \cdot r^{-2} \\ &= sr^{-4} \end{aligned}$$

$s \sim sr^{-2} \sim sr^{-4} \sim sr^{-6} \sim sr^{-8} \sim \cdots$

Since $n$ is odd, the conjugacy class of $s$ is $\{s, sr, \ldots, sr^{n-1}\}$

$\neq C_{D_{2n}}(s) = 2$ $C_{D_{2n}}(s) = \{e, s\}$



$D_{2n}, n$ even

$\{r, r^{-1}\}, \{r^2, r^{-2}\}, \ldots, \left\{r^{\frac{n-2}{2}}, r^{\frac{n+2}{2}}\right\}, \left\{r^{\frac{n}{2}}\right\}$ are conjugacy classes.

$C_{D_{2n}}(s) = \left\{e, s, r^{\frac{n}{2}}, sr^{\frac{n}{2}}\right\}$

$\{s, sr^2, sr^4, \ldots, sr^{n-2}\}$ and $\{sr, sr^3, \ldots, sr^{n-1}\}$ are conjugacy classes

Ex $S_n$

conjugacy classes are given by cycle structures $[\sigma] :=$ conjugacy classes of $\sigma, \sigma \in S_n$

$S_4$ : $[e]$, $[(1\ 2)]$, $[(1\ 2\ 3)]$, $[(1\ 2)(3\ 4)]$, $[(1\ 2\ 3\ 4)]$
sizes: 1 6 8 3 6

Theorem[Class Equation] Let $G$ be a finite group. Let $g_1, g_2, \ldots, g_m$ be a complete, non-repetitive list of conjugacy classs representatives that are not central. Then

$$\#G = \#(Z(G)) + \sum_{i=1}^{n} [G : C_G(g_i)]$$

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Proof Apply LOIS to action of $G$ in $G$ by conjugation

Ex (1) $D_{2n}, n$ odd

$$2n = 1 + \underbrace{2 + 2 + \ldots + 2}_{\frac{n-1}{2}} + n$$

(2) $S_{2n}, n$ even:

$$2n = 2 + \underbrace{2 + \cdots + 2}_{\frac{n-2}{2}} + \frac{n}{2} + \frac{n}{2}$$

(3) $S_4 : 24 = 1 + 6 + 8 + 3 + 6$

Theorem If $\#G = p^r$, $p =$ prime, then $Z(G) \neq \{e\}$.

Pf: $p^r = \#G = \#Z(G) + \sum_{i=1}^{m} [G : C_G(g_i)]$

$C_G(g_i) \lneqq G \Rightarrow [G : C_G(g_i)]$ is divisible by $p$, using Lagrange.

$\therefore p \mid \#Z(G)$ and $\#Z(G) \geq 1$.

$\therefore\ \#Z(G) \geq p$. □