MATH 817 Notes

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(e.g., if $p \mid \#G$, p is prime, G is simple $\Rightarrow$ G cannot act non-trivially on a set of n elements with $n < p$)

Products and Semi-direct products

Let $G_1, G_2, \ldots, G_n$ be a list of groups. Then $G := G_1 * G_2 * \cdots * G_n$ is a group under

$$(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) := (x_1y_1, x_2y_2, \ldots, x_ny_n)$$

with identity element $(e, \ldots, e)$ and

$$(x_1, \ldots, x_n)^{-1} = (x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})$$

G = "product" of $G_1, G_2, \ldots, G_n$.

Notation: $G = \prod_{i=1}^{n} G_i$

G has the following properties:

① $\tilde{G}_i := \left\{ (e, e, \ldots, e, \underset{i^{\text{th}}}{x}, e, \ldots, e) \mid x \in G_i \right\} \leq G$.

② $G_i \cong \tilde{G}_i$ via $x \mapsto (e, \ldots, e, \underset{i}{x}, e, \ldots, e)$

③ If $v \in \tilde{G}_i$ and $w \in \tilde{G}_j$ for $i \neq j$, $v \cdot w = w \cdot v$

④ $\tilde{G}_i \trianglelefteq G, \forall i$ $[N_G(\tilde{g}_i) \supseteq \tilde{G}_i, N_G(\tilde{G}_i) \supseteq \tilde{G}_j, \forall j, \langle \tilde{G}_1, \ldots, \tilde{G}_n \rangle = G]$

⑤ The projection function $\pi_i : G \to G_i$, defined by $\pi_i(x_1, \ldots, x_n) = x_i$, is a surjective group homomorphism.

⑥ For any group H, there is a bijection

$$\{\varphi : H \to G \mid \varphi \text{ is a group homomorphism}\} \underset{\text{onto}}{\overset{\text{1-1}}{\longleftrightarrow}} \{(\varphi_1, \varphi_2, \ldots, \varphi_n) \mid \varphi_i : H \to G_i \text{ is a group homomorphism}\}$$

given by
$$\begin{array}{rcl} \varphi & \mapsto & (\pi_1 \circ \varphi, \ldots, \pi_n \circ \varphi) \\ (\text{the map } h \mapsto (\varphi_1(h), \ldots, \varphi_n(h))) & \leftarrow\!\shortmid & (\varphi_1, \ldots, \varphi_n) \end{array}$$

$n = 2$: $G = G_1 \times G_2$

Let $H = \tilde{G}_1 = \{(x, e) \mid x \in G_1\}$

$K = \tilde{G}_2 = \{e, y) \mid y \in G_2\}$

Then

- $H \trianglelefteq G$
- $K \trianglelefteq G$
- $H \cap K = \{e\}$

- $HK = G$

Prop If H is a group and H, K are subgroups so that $H \trianglelefteq G$, $K \trianglelefteq G$ and $H \cap K = \{e\}$, then the function $\varphi : H \times K \to G$ by $\varphi(h,k) = h \cdot k$ is an injective group homomorphism and thus $H \times K = HK \leq G$.