MATH 817 Notes

JD Nir
jnir@huskers.unl.edu
www.math.unl.edu/∼jnir2/817.html
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Recall: If G is a group and $\#G = 30$, then

- $\exists a \in G$ $|a| = 15$ and so $N := \langle a \rangle \trianglelefteq G$.
- $\exists b \in G$ such that $|b| = 2$. It exists by Cauchy and is not in $\langle a \rangle$ by Lagrange.
- Let $\rho : G \to \mathrm{Aut}(N)$ be the group homomorphism associated to the conjugation action of G on N. Then
$$\mathrm{Aut}(N) \underset{(*)}{\cong} (\mathbb{Z}/15)^\times$$
and $\rho(b)$ corresponds under $(*)$ to $j = 1, 4, 11$ or 14. ($|b| = 2 \Rightarrow |\rho(b)| = 1$ or 2)

So, since $(\mathbb{Z}/15)^\times \cong \mathrm{Aut}(N)$ via $j \mapsto (a \mapsto a^j)$ $\therefore$ $bab^{-1} = bab = a^j$, $j = 1$ or 4 or 11 or 14.

E.g.

- If $j = 1$, $ba = ab$ $\therefore$ $G \cong \mathbb{Z}/30, G = \langle ab \rangle$
- If $j = 14 \equiv -1(15), bab^{-1} = a^{14} = a^{-1}$. $\therefore G \cong D_{30}$

Summary: $\#G = 30 \Rightarrow G = \langle a, b \rangle, |a| = 15, |b| = 2, bab = a^j, j = 1, 4, 11, 14$. And if $j = 1, G \cong \mathbb{Z}/30$ and if $j = 14, G \cong D_{30}$.

Theorem [We won't prove it] Given symbols $x_1, x_2, \ldots, x_n$ and "words" $R_1, \ldots, R_m$ where each R_i has the form $x_{i_1}^{e_1} \cdot x_{i_2}^{e_2} \cdots x_{i_\ell}^{e_\ell}, i_j \in \{1, \ldots, n\}$ and $e_j \in \mathbb{Z}$, $\exists$ a group $U = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_m \rangle$ such that

(1) U is generated by $x_1, \ldots, x_n \in U$

(2) $R_i = e$ in U

(3) Given any group G and elements $g_1, \ldots, g_n \in G$ so that $R_i(g_1, \ldots, g_n) = e$ in G, then $\exists!$ group homomorphism $\varphi : U \to G$ with $\varphi(x_i) = g_i$.

e.g. $\langle x, y \mid x^{15}, y^2, yxyx^{-4} \rangle$
$$\Updownarrow$$
$$yxy^{-1} = x^4 \Leftrightarrow yx = x^4 y^{-1}$$

$\#U \leq 30$: Since $yx = x^4 y^{-1}$, every element in U can be written as $x^i y^j$, $i, j \in \mathbb{Z}$. Since $x^{15} = e, y^2 = e$, we can take $0 \leq i \leq 14$ and $0 \leq j \leq 1$ ✓

We can't yet show $\#U = 30$.

$G = \langle a, b \rangle$ By Theorem, $\exists$ group homomorphism $\varphi : U \to GGG$, where $U = \langle x, y \mid x^{15}, y^2, yxyx^{-j} \rangle$ and $\varphi(x) = a$ and $\varphi(y) = b$.

- φ is onto because a, b generate H and are in $\mathrm{im}\, \varphi$.
- $\#U \leq 30$

JD Nir
jnir@huskers.unl.edu

- $\#G = 30$

$\therefore$ φ is an isomorphism (pigeonhole).

$G \cong \langle x, y \mid x^{15}, y^2, yxyx^{-j} \rangle$

$\varphi : U \to G$

$\operatorname{im} \varphi \supseteq \langle a, b \rangle$

Since $a, b \in \operatorname{im} \varphi$, $\langle a, b \rangle \subseteq \operatorname{im} \varphi$ since $\operatorname{im} \varphi \leq G$.

$\langle x, y \mid x^{15}, y^2, yxyx^{-1} \rangle$
$\langle x, y \mid x^{15}, y^2, yxyx \rangle$ have order 30.

$j = 4$ and 11 are unsettled.

Prop $\#G = p^2 q$, p and q are primes $\Rightarrow$ G is solvable.

Pf ETS G is not simple, since we know groups of order p, pq, p^2 are solvable. (If $N \trianglelefteq G, N, G/N$ are solvable $\Rightarrow$ G is solvable.)

- $p = q \Rightarrow Z(G) \neq \{e\}$ ✓
- $p > q$: $n_p \equiv 1 (\text{mod } p)$ and $n_p \mid q \Rightarrow n_p = 1$. $\therefore$ $\exists P \triangleleft G, \#P = p^2$.
- $p < q$ $n_q \equiv 1 (\text{mod } q)$ and $n_q \mid p^2$. $\therefore n_q = 1$ or p or p^2.
  - If $n_1 = 1$, done since $\exists Q \triangleleft G, \#Q = q$.
  - $n_q = p$ is not possible ($p \not\equiv 1 (\text{mod } q)$)
  - $n_q = p^2$. So, $p^2 \equiv 1 (\text{mod } q) \Rightarrow q \mid p^2 - 1 \Rightarrow q \mid (p-1)(p+1) \Rightarrow q \mid q - 1$ or $q \mid p + 1$. Since $p < q$, $q = p + 1$. $\therefore$ $p = 2$ and $q = 3$ and $\#G = 12$.

  So, $n_3 = 4 = \# \operatorname{Syl}_3(G)$.

  New trick G acts on $\operatorname{Syl}_3(G)$. This gives a group homomorphism $\rho : G \to S_4$.

  $\operatorname{im} \rho \neq \{e\}$: if $\operatorname{im} \rho = \{e\}$, then the action of G on $\operatorname{Syl}_3(G)$ would be trivial. This contradicts Sylow #2.

  $\therefore \ker \rho \triangleleft G$.

  If $\ker \rho = \{e\}, G \cong \operatorname{im} \rho \leq S_4$, S_4 is known to be solvable.

  $\therefore$ G is solvable.

  ($G \cong A_4$)