MATH 817 Notes

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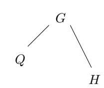
<u>Pf of Lemma</u>: ($\supseteq$) is obvious because $Q \subseteq N_G(Q)$.

($\subseteq$) $H := P \cap N_G(Q)$. $Hu \subseteq P$ is clear. Enough To Show $H \subseteq Q$. Since $H \subseteq N_G(Q)$, $H \cdot Q \leq G$ and in fact (2nd Isomorphism Theorem)

$$H \cdot Q / Q \cong H / H \cap Q.$$



In particular, $\#(H \cdot Q) = \frac{\#H \cdot \#Q}{\#(H \cap Q)}$. $H \leq P$, $\#P = p^j$, $Q \in \mathrm{Syl}_p(G) \Rightarrow \#(H \cdot Q) = p^m$, some m. But $Q \leq H \cdot Q$ and $Q \in \mathrm{Syl}_p(G) \Rightarrow Q\ H \cdot Q \Rightarrow H \subseteq Q$. □

<u>Pf of ③</u> G acts on $\mathrm{Syl}_p(G)$. By ①, $\exists p \in \mathrm{Syl}_p(G)$. P acts on $\mathrm{Syl}_P(G)$ too. Consider the orbits of the action of P on $\mathrm{Syl}_p(G)$. If $y \in P$, $yPy^{-1} = P$. So, $\{P\}$ is an orbit for the action of P on $\mathrm{Syl}_P(G)$. It suffices to prove that every other orbit $\mathcal{O}$, besides $\{P\}$, for the action of P on $\mathrm{Syl}_p(G)$ has size divisible by p. Pick such an $\mathcal{O}$. Let $Q \in \mathcal{O}$. The stabilizer of the action of P for Q is

$$\left\{y \in P \mid yQy^{-1} = Q\right\} = P \cap N_G(Q) \stackrel{\text{Lemma}}{=} P \cap Q < P, \text{ since } P \neq Q \text{ and } \#P = \#Q.$$

By LOIS: $\#\mathcal{O} = [P : P \cap Q]$ and $p \mid [P : P \cap Q]$. This proves ③.