MATH 817 Notes

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p prime, an e. a. p. g. is an abelian group $(V, +)$ so that $p \cdot v = 0\ \forall v \in V$.

Prop (1) If $(V, +)$ is an e. a. p. g. then V is a $\mathbb{Z}/p$ vector space with scalar multiplication defined as $\overline{m} \cdot v := mv = \underbrace{v + \ldots + v}_{m}$

(2) If V, W are two e.a.p.g.'s, a function $\rho : V \to W$ is a group homomorphism iff it is a $\mathbb{Z}/p$-linear transformation.

Pf Easy. For (2), $\Leftarrow$ is immediate. For $\Rightarrow$,

$$\rho(\overline{m} \cdot v) = \rho\left(\overbrace{v + \ldots + v}^{m}\right) = \overbrace{\rho(v) + \ldots + \rho(v)}^{m} = \overline{m} \cdot \rho(v)$$

Corollary If V is an e.a.p.g. and $\#V < \infty$, then

(1) $V \cong \overbrace{\mathbb{Z}/p \times \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p}^{n}$, some n

(2) $\mathrm{Aut}(V) \cong GL_n(\mathbb{Z}/p) := \{n \times n \text{ matrices } A \text{ with entries in } \mathbb{Z}/p \text{ such that } \det A \neq 0\}$

Prop $\#GL_n(\mathbb{Z}/p) = (p^n - 1)(p^n - p)(p^n - p^2) \cdots (p^n - p^{n-1})$

sketch of proof: Let $[\vec{v_1}|\vec{v_2}| \cdots |\vec{v_n}]$ be any $n \times n$ matrix with entries in $\mathbb{Z}/p$. So, $\vec{v_i}$ is a column vector with n entries.

$$\begin{aligned} A \in GL_n(\mathbb{Z}/p) &\Leftrightarrow \{\vec{v_1}, \ldots, \vec{v_n}\} \text{ is a linearly independent set} \\ &\Leftrightarrow \vec{v_1} \neq 0 \\ &\quad \vec{v_2} \notin \text{span}\{\hat{v}_1\} \\ &\quad \vec{v_3} \notin \text{span}\{\hat{v}_1, \vec{v_2}\} \\ &\quad \vdots \\ &\quad \vec{v_{j+1}} \notin \text{span}\{\vec{v_1}, \ldots, \vec{v_j}\}, j = 1, \ldots, n-1 \end{aligned}$$

If $\vec{v_1}, \ldots, \vec{v_j}$ are linearly independent, then

$$\text{span}\{\vec{v_1}, \ldots, \vec{v_j}\} \cong \underbrace{\mathbb{Z}/p \times \cdots \times \mathbb{Z}/p}_{j}$$

and thus $|\,\text{span}\{\vec{v_1}, \ldots, \vec{v_j}\}| = p^j$

So, there are $p^n - 1$ choices for $\vec{v_1}$
$p^n - p$ choices for $\vec{v_2}$
$\vdots$
$p^n - p^j$ choices for $\vec{v_{j+1}}$ $\square$

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Application If $\#G = 45$ and $\exists N \trianglelefteq G$ such that $\#N = 9$, then G is abelian.

| Note By Sylow Theory, every group of order 45 has a normal subgroup of order 9.
↓ $\therefore\ \#G = 45 \Rightarrow G$ abelian.

Pf: Since $N \trianglelefteq G$, G acts on N via conjugation and so $\exists \rho : G \to \mathrm{Aut}(N)$ and $\ker \rho = C_G(N)$.

Claim: $C_G(N) = G$.

Pf: $\#N = 9 \Rightarrow N$ is abelian

Case 1 N is cyclic.

Then $\mathrm{Aut}(N) \cong \mathrm{Aut}(\mathbb{Z}/9) \cong (\mathbb{Z}/9)^\times$ and so $\#\,\mathrm{Aut}(N) = 6$

But $G/C_G(N) \cong \mathrm{im}\,\rho \leq \mathrm{Aut}(N) \Rightarrow \#\,\mathrm{im}\,\rho \mid \#Aut(N)$

N abelian $\Rightarrow N \leq C_G(N) \Rightarrow \#G/C_G(N) = 1$ or 5, since $[G : N] = 5$.

Since $5 \nmid 6$, $\#G/C_G(N) = 1$ in this case.

Case 2 N is not cyclic.

So, $|n| = 1$ or 3 $\forall n \in N$.

$\Rightarrow N$ is a e.a.3g. $\Rightarrow N \cong \mathbb{Z}/3 \times \mathbb{Z}/3 \Rightarrow \mathrm{Aut}(N) \cong GL_2(\mathbb{Z}/3)$

$\Rightarrow \#Aut(N) = (3^2 - 1)(3^2 - 3) = 48.$ $5 \nmid 48 \Rightarrow \#\,\mathrm{im}\,\rho = 1 \Rightarrow C_G(N) = G$.

This proves the claim.

Pick any $x \in G \setminus N$. Since $[G : N] = 5, \langle N, x\rangle = G$.

Since N is abelian and $x \in C_G(N)$, $\langle N, x\rangle$ is ableian. $\square$

Def G is a finite group, $\#G = p^\ell \cdot m, p \nmid m, p$ is prime ($\ell = 0$ is allowed)

A Sylow p-subgroup of G is a subgroup $P \leq G$ such that $\#P = p^\ell$.

Ex (1) $\#G = 45, N \leq G, \#N = 9 \Rightarrow N$ is a Sylow 3-subgroup of G.

(2) $G = S_5$ $p = 5$ $\#G = 5 \cdot 24$ $(\ell = 1)$

$H_1 = \langle (1\ 2\ 3\ 4\ 5)\rangle$ is a Sylow 5-subgroup of S_5

$H_2 = \langle (2\ 1\ 3\ 4\ 5)\rangle$ is another

$H_3 = \langle (1\ 2\ 3\ 5\ 4)\rangle$ is another

$\vdots$

H_6

Notes In ex. (2)

- $H_1, \ldots, H_6$ are all conjugate to each other
- $6 \equiv 1 (\mathrm{mod}\ 5)$
- $6 \mid 24$

Theorem p prime, G finite group, $\#G = p^\ell \cdot m, p \nmid m$. Let $\mathrm{Syl}_P(G)$ = set of all Sylow p-subgroups of G and $n_p = \# \mathrm{Syl}_p(G)$. Then

(1) $n_p \geq 1$

(2) G acts transitively on $\mathrm{Syl}_p(G)$ by conjugation: if $P \in \mathrm{Syl}_p(G), xPx^{-1} \in \mathrm{Syl}_p(G)\ \forall x \in G$. If $P, Q \in \mathrm{Syl}_p(G)$, then $Q = xPx^{-1}$, some $x \in G$.

(3) $n_p \equiv 1 (\mathrm{mod}\ p)$

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(4) $n_p = [G : N_G(P)]$ for any $P \in \mathrm{Syl}_p(G)$

(5) $n_p \mid m$