MATH 817 Notes

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No class Friday or Monday

- Next problem set?

<u>Automorphism groups</u>

So far:

$$\mathrm{Aut}(\mathbb{Z}/n) \cong (\mathbb{Z}/n)^\times, n \geq 2$$

$\mathrm{Aut}(\mathbb{Z}) \cong (\{\pm 1\}, \cdot)$

In general, $\exists$ group homomorphism $\rho : G \to \mathrm{Aut}(G)$ by $\rho(g) = \varphi_g$, where $\varphi_g : G \to G$ is $\varphi_g(x) = gxg^{-1}$.

$\ker(\varphi) = \left\{g \mid gxg^{-1} = x \ \forall x\right\} = Z(G)$

$$\therefore \ G/Z(G) \cong \underset{\text{"inner automorphisms"}}{\underset{\|}{\mathrm{Inn}(G)}} := \mathrm{im}(\rho) \leq \mathrm{Aut}(G)$$

<u>Note</u>: $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$

$$\frac{\mathrm{Aut}(G)}{\mathrm{Inn}(G)} =: \mathrm{Out}(G)$$

$\mathrm{Inn}(G) = \{\alpha \mid \alpha \in \mathrm{Aut}(G) \text{ and } \alpha = \varphi_g, \text{some } g\}$

<u>Ex</u> $\begin{aligned} G &= D_{10} \\ &= \langle r, s \mid r^5, s^2, srs = r^{-1}\rangle \end{aligned}$

$Z(D_{10}) = \{e\}$

$\therefore \rho : D_{10} \xrightarrow{\cong} \mathrm{Inn}(D_{10}) \leq \mathrm{Aut}(D_{10})$

$\alpha \in \mathrm{Aut}(D_{10}) \Rightarrow \begin{aligned} \alpha(r) &= r^i, 1 \leq i \leq 4 \\ \alpha(s) &= sr^j, 0 \leq j \leq 4 \end{aligned}$

α is uniquely determined by i, j:

$\therefore \ \#Aut(D_{10} \leq 20$

$$\begin{aligned} &\rho(r) = \varphi_r : \varphi_r(r) = r && i = 1 \\ &\varphi_r(s) = rsr^{-1} = sr^{-1} = sr^3 && j = 3 \end{aligned}$$

e.g. $$\begin{aligned} &\varphi_s\text{: } \varphi_s(r) = srs = r^{-1} = r^4 && i = 4 \\ &\varphi_s(s) = s && j = 0 \end{aligned}$$

$$\begin{aligned} &\varphi_{r^2} = \varphi_r \circ \varphi_r \ \ \varphi_{r^2}(r) = r && i = 1 \\ &\varphi_{r^2}(s) = \varphi_r(\varphi_r(s)) = \varphi_r(sr^3) = \varphi_r(s)\varphi_r(r^3) = sr^3 \cdot r^3 = sr && j = 1 \end{aligned}$$

Let i, j be abitrary, $1 \leq i \leq 4$ and $0 \leq j \leq 4$

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Let $r' = r^i$

Let $s' = sr^j$

$|r'| = 5$

$|s'| = 2$

$$\begin{aligned} s'r's' &= sr^j r^i sr^j \\ &= s^2 r^{-j} r^{-i} r^j \\ &= r^{-i} \\ &= (r')^{-1} \end{aligned}$$

$\therefore r', s'$ satisfy same relations as r, s

$\Rightarrow \exists$ homomorphism $\alpha : D_{10} \to D_{10}$

$\alpha(r) = r'$, $\alpha(s) = s'$ and α is onto

$\therefore\ \alpha \in \text{Aut}(D_{10})$

$\therefore\ \#Aut(D_{10}) = 20$

e.g. $\exists$ automorphism $\alpha : D_{10} \to D_{10}$, $\alpha(r) = r^2$, $\alpha(s) = s\ \alpha \in \text{Aut}(G)$

Claim $\alpha \notin \text{Inn}(G)$

$\varphi_r(r) = r$ and $\varphi_s(r) = r^{-1} \Rightarrow$ any product of φ_r, φ_s sends r to r or r^{-1}

$\therefore\ \beta \in \text{Inn}(D_{10})$, $\beta(r) = r^{\pm 1}$.

Q: Does α have geometric meaning?

Ex $Z(S_n) = \{e\}$, $n \geq 3$

$\therefore \rho : S_n \to \text{Aut}(S_n)$ is 1-1 and $S_n \cong \text{Inn}(S_n)$

Theorem: $\text{Inn}(S_n) = \text{Aut}(S_n)$ for $n \neq 6$

Note $[\text{Aut}(S_6) : \text{Inn}(S_6)] = 2$

Ex A_5 $Z(A_5)\,\{e\}$

$\rho : A_5 \to \text{Aut}(A_5)$ is 1-1

$A_5 \cong \text{Inn}(A_5) \leq \text{Aut}(A_5)$

Party question: Is $\text{Out}(A_5)$ trivial? No

Observe: Given G, if $G \trianglelefteq L$, $\exists$ map $L \to \text{Aut}(G)$ (L acts on conjugation of G).

$A_5 \trianglelefteq S_5$. Pick (1 2).

Let $\alpha : A_5 \to A_5$ be conjugation by $(1\ 2) = \tau$

$\alpha((1\ 2\ 3)) = \tau(1\ 2\ 3)\tau^{-1} = (\tau(1)\ \tau(2)\ \tau(3)) = (2\ 1\ 3)$

$\alpha((1\ 2\ 3\ 4\ 5)) = (2\ 1\ 3\ 4\ 5)$ and $(1\ 2\ 3\ 4\ 5)$ is not conjugate to $(2\ 1\ 3\ 4\ 5)$ in A_5.

$\alpha \in \text{Inn}(A_5) \Rightarrow \alpha = \varphi_\sigma, \sigma \in A_5$

$(2\ 1\ 3\ 4\ 5) = \varphi_\sigma((1\ 2\ 3\ 4\ 5)) = \sigma(1\ 2\ 3\ 4\ 5)\sigma^{-1} \sim (1\ 2\ 3\ 4\ 5)$

$\therefore\ \alpha \notin \text{Inn}(A_5)$

<u>Note</u> $\mathrm{Aut}(A_5) \xleftarrow{\cong} S_5$, via the map sending $\tau \in S_5$ to conjugation by τ in A_5

<u>Def</u>: Let p be a prime. An <u>elementary abelian p-group</u> (eapg) is an abelian group $(V, +)$ so that $p \cdot v = 0\ \forall v \in V$.

(If $(V, \cdot), v^p = 1\ \forall v$)

<u>Fact</u> if V is eapg, then V is a vector space over the field $\mathbb{Z}/p$

($\mathbb{Z}/p$ is a field: If $\overline{m} \neq \overline{0}$ in $\mathbb{Z}/p$,

$$\begin{aligned} p \nmid m &\Rightarrow \gcd(p, m) = 1 \\ &\Rightarrow ap + bm = 1, a, b \in \mathbb{Z} \\ &\Rightarrow \overline{b} \cdot \overline{m} = \overline{1}) \end{aligned}$$

The rule for scaling is: If $\overline{m} \in \mathbb{Z}/p$, $v \in V$,

$$\overline{m} \cdot v := mv = \overbrace{v + \ldots + v}^{m}$$

$$\begin{aligned} (\overline{m} = \overline{n} \Rightarrow m &= n + \ell \cdot p \\ \Rightarrow mv &= (n + \ell p)v \\ &= nv + \ell pv \\ &= nv. \end{aligned}$$

$\therefore\ V \cong$ a direct sum of copies of $\mathbb{Z}/p$