MATH 817 Notes

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$N \trianglelefteq G$

G has a composition series $\Rightarrow$ N and ${}^{G}/_{N}$ do too.

Pf Let $\{e\} = M_0 \trianglelefteq M_1 \trianglelefteq \ldots \trianglelefteq M_\ell = G$ be a composition series.

$\{e\} \trianglelefteq N \trianglelefteq G$ is a normal series

Recall: Any two normal series have equivalent refinements.

So, $\{e\} \trianglelefteq N \trianglelefteq G$ admits a refinement

$$\{e\} \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq \underset{\substack{\| \\ N}}{N_j} \trianglelefteq N_{j+1} \trianglelefteq \ldots \trianglelefteq N_m = G$$

tht is equivalent to some refinement of the composition series.

So $\forall i$, ${}^{N_i}/_{N_{i-1}}$ is either simple or trivial.

Upon deleting the repetitions we obtain:

$\exists$ a composition series of G that "passes through" N.

G has 3 conjugacy classes, $\#G < \infty$.

$\#G = \#Z(G) + \sum_{i=1}^{r} [G : C_G(g_i)]$ $g_1, \ldots, g_r$ are non-central representatives of conjugacy classes.

$\#Z(G) = 1$, $r = 2$

$\#G = 1 + [G : C_G(x)] + [G : C_G(y)]$

You may use: If $p \mid \#G$, then $\exists$ element of order p.

Tip: Show $\#G = p \cdot q$, p and q are distinct primes.

$\#G = p^r$, $N \trianglelefteq G, N \neq \{e\}$

(a) $N \cap Z \neq \{e\}$

(b) Prove $\exists H \leq N$ such that $[N : H] = p$

Key fact:

If $G = \langle S \rangle, \#S < \infty$, $\#L < \infty$ then there are a finite # of group homomorphisms from G to L.

$(L = H_n)$

If $H \leq G$ and $[G : H] = n$ then G acts on left cosets of H and hence $\exists \rho : G \to S_n$.

Tip $\ker \rho \subseteq H$

Example Groups of order $p \cdot q$.

- If $p = q$, G is abelian.
- Say $p < q$ and G is not abelian.
  Then $Z(G) = \{e\}$
  $\#G = 1 + \sum_{i=1}^{r} [G : C_G(g_i)]$
  $[G : C_G(g_i)] = p$ or q
  $\Rightarrow \#C_G(g_i) = q$ or p
  But $\langle g_i \rangle \subseteq C_G(g_i)$
  $\therefore\ \#C_G(g_i) = q \Rightarrow |g_i| = q$ and $\langle g_i \rangle = C_G(g_i)$
  $\#G = 1 + ap + bq$, $ap =$ number of elements of order q, $bq =$ number of elements of order p
  Note $a \neq 0$ $b \neq 0$
  Let x be an element of order q.
  Let y be an element of order q.
  So, $\langle x \rangle = C_G(x)$
  Also, $\langle x, y \rangle = G$
  $[G : \langle x \rangle] = p \Rightarrow \langle x \rangle \trianglelefteq G$
  $\exists$ homomorphism $\rho : G \to \mathrm{Aut}(\langle x \rangle) \cong \mathrm{Aut}(\mathbb{Z}/_q) = (\mathbb{Z}/_q)^\times$
  $\rho(g) = \varphi_g, \varphi_g(x) = gxg^{-1}$
  $\rho(x) =$ identity
  $\rho(y) = \varphi_y, \varphi_y(x) = yxy^{-1} = x^j$, some j
  $|y| = p \Rightarrow |\rho(y)| = 1$ or p
  If $|\rho(y)| = 1$, G is abelian
  If $|\rho(y)| = p$,
  $yxy^{-1} = x^j$
  $1 \leq j \leq q - 1$
  $\varphi_y^p(x) = ((x^j)^j)^{\cdot^{\cdot^{\cdot^j}}} = x^{j^p}$
  $j^p \equiv 1 (\mathrm{mod}\ q)$
  $G = \langle x, y \mid x^q, y^p, yxy^{-1} = x^j \rangle$