MATH 107-253 Recitation 10 Solutions

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1: Integrate each of the following without using an integration table.

$$\text{(a)} \int \frac{1}{\sqrt{1-u^2}}\ du \quad \text{(b)} \int \frac{1}{1-u^2}\ du \quad \text{(c)} \int \frac{u}{\sqrt{1-u^2}}\ du \quad \text{(d)} \int \frac{u}{1-u^2}\ du.$$

Simplify your answer as much as possible.

(a) You may recognize this integral as the derivative of $\arcsin(x)$. While we'll hope to get this answer, let's use a trig substitution to find it.

Because we have the form $a^2 - x^2$ (where $a = 1$ and $x = u$) we should use

$$\sin\theta = \frac{u}{1} = u.$$

Substituting gives

$$\int \frac{1}{\sqrt{1-u^2}}\ du = \int \frac{1}{\sqrt{1-(\sin\theta)^2}}\ du.$$

Notice that we have mixed variables: our integral is in terms of θ but our differential is in terms of du. So we'll have to find an equation for du in terms of $d\theta$.

$$\frac{d}{du}[\sin\theta] = \frac{d}{du}[u] \implies \cos\theta \cdot \frac{d\theta}{du} = 1 \implies \cos\theta\ d\theta = du$$

So now we can substitute and get:

$$\int \frac{1}{\sqrt{1-(\sin\theta)^2}}\ du = \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\ d\theta.$$

Now we use the trig identity

$$\sin^2\theta + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \sin^2\theta$$

to get

$$\int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\ d\theta = \int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\ d\theta$$
$$= \int \frac{\cos\theta}{\cos\theta}\ d\theta$$
$$= \int 1\ d\theta$$
$$= \theta + C$$

Now we need to change our answer from being in terms of θ to being in terms of u. We have

$$\sin\theta = u \implies \arcsin(\sin\theta) = \arcsin u \implies \theta = \arcsin u$$

so

$$\int \frac{1}{\sqrt{}}\ du = \arcsin(u) + C.$$

As expected, both methods give us the same answer.

Now we can use our partial fraction decompostion to write

$$\int \frac{1}{(1+u)(1-u)} \, du = \int \frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u} \, du$$
$$= \frac{1}{2} \int \frac{1}{1+u} \, du + \frac{1}{2} \int \frac{1}{1-u} \, du$$

While these are almost of the form $\int \frac{1}{x} \, dx = \ln|x|$, they aren't quite that simple. We may be tempted to skip the substitution step, but that would be dangerous in this case for reasons we'll see in a moment. Because our variable is u, we'll do a w-substitution.

For the first substitution we get $w = 1 + u$. Then $dw = du$ and

$$\frac{1}{2} \int \frac{1}{1+u} \, du = \frac{1}{2} \int \frac{1}{w} \, dw = \frac{1}{2} \ln|w| + C_1 = \frac{1}{2} \ln|1 + u| + C_1.$$

For the second substitution we use $w = 1 - u$. Then $dw = -du$. Note the minus sign! If we'd skipped the substitution step, it's possible we would have missed it!

$$\frac{1}{2} \int \frac{1}{1-u} \, du = \frac{1}{2} \int \frac{1}{w}(-dw) = -\frac{1}{2} \int \frac{1}{w} \, dw = -\frac{1}{2} \ln|w| + C_2 = -\frac{1}{2} \ln|1 - u| + C_2$$

Now we can combine those two equations. We can also combine the constants into one constant.

$$\frac{1}{2} \int \frac{1}{1+u} \, du + \frac{1}{2} \int \frac{1}{1-u} \, du = \frac{1}{2} \ln|1 + u| - \frac{1}{2} \ln|1 - u| + C$$

Because the question asks us to simplify as much as possible, we should use some of the laws of logs:

$$\frac{1}{2} \ln|1 + u| - \frac{1}{2} \ln|1 - u| + C = \frac{1}{2}(\ln|1 + u| - \ln|1 - u|) + C$$
$$= \frac{1}{2}\left(\ln\left|\frac{1+u}{1-u}\right|\right) + C$$
$$= \frac{1}{2} \ln\left|\frac{1+u}{1-u}\right| + C$$
$$= \ln\left|\frac{1+u}{1-u}\right|^{1/2} + C$$
$$= \ln\sqrt{\left|\frac{1+u}{1-u}\right|} + C$$

This gives us a final answer of

$$\int \frac{1}{1-u^2} \, du = \ln\sqrt{\left|\frac{1+u}{1-u}\right|} + C.$$

This is a pretty complicated answer. Let's make sure it's right by taking the derivative.

$$\frac{d}{du}\left[\ln\sqrt{\left|\frac{1+u}{1-u}\right|} + C\right] = \frac{d}{du}\left[\ln\sqrt{\left|\frac{1+u}{1-u}\right|}\right] + \frac{d}{du}[C]$$
$$= \frac{1}{\sqrt{\left|\frac{1+u}{1-u}\right|}} \cdot \frac{d}{du}\left[\sqrt{\left|\frac{1+u}{1-u}\right|}\right] + 0$$

So now we can write

$$\int \frac{\sin\theta}{\sqrt{1 - (\sin\theta)^2}}\, du = \int \frac{\sin\theta}{\sqrt{1 - \sin^2\theta}} \cdot \cos\theta\; d\theta.$$

We can use the trig identity $\cos^2\theta = 1 - \sin^2\theta$ again to get

$$\int \frac{\sin\theta}{\sqrt{1 - \sin^2\theta}} \cdot \cos\theta\; d\theta = \int \frac{\sin\theta\cos\theta}{\sqrt{\cos^2\theta}}\; d\theta$$

$$= \int \frac{\sin\theta\cos\theta}{\cos\theta}\; d\theta$$

$$= \int \sin\theta\; d\theta$$

$$= -\cos\theta + C$$

Now we solve for θ:

$$\sin\theta = u \implies \arcsin(\sin\theta) = \arcsin(u) \implies \theta = \arcsin(u)$$

and get

$$-\cos\theta + C = -\cos(\arcsin(u)) + C.$$

We can simplify this answer further. Remember that the definition of $\arcsin(u)$ is "the angle whose sine is u." We can look at what that angle must look like. Lets call that angle φ. Remember that the sine of an angle is the ratio of the side of the triangle opposite the angle over the hypotenues of the triangle. Then we can use the Pythagorean Theorem to fill in the last side of the triangle.
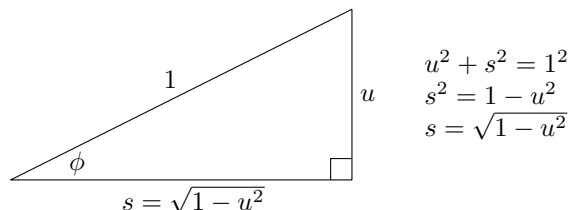
Now let's return to simplifying $\cos(\arcsin(u))$. Since we set $\arcsin(u) = \varphi$, we can look at the triangle to find

$$\cos(\arcsin(u)) = \cos(\varphi) = \sqrt{1 - u^2}$$

giving us a final answer of

$$\int \frac{u}{\sqrt{1 - u^2}}\, du = \sqrt{1 - u^2} + C.$$

Pretty amazing! Even though we used a trig substitution, the final answer has nothing to do with trig functions. Just to make sure we got it right, let's take the derivative to check.

$$\frac{d}{du}\left[\sqrt{1 - u^2} + C\right] = \frac{d}{du}\left[\sqrt{1 - u^2}\right] + \frac{d}{du}[C]$$

$$= \frac{d}{du}[(1 - u^2)^{1/2}] + 0$$

$$= -\frac{1}{2}\cdot(1 - u^2)^{-1/2}\cdot\frac{d}{du}[1 - u^2]$$

$$= -\frac{1}{2(1 - u^2)^{1/2}}\cdot(-2u)$$

$$= -\frac{-2u}{2\sqrt{1 - u^2}}$$

$$= -\frac{u}{}$$

p. 378 #7: Find

$$\int_0^1 \ln x \ dx$$

or explain why it does not converge.

Note that $\ln(0)$ is not defined so we need to treat this as an indefinite integral. So if this integral does converge we will have

$$\int_0^1 \ln(x) \ dx = \lim_{a \to 0^+} \int_a^1 \ln(x) \ dx.$$

Now let's look at the integral. We don't have an integration rule that solves this integral, but we can solve it with integration by parts.

While we don't know how to take the integral of $\ln x$, it is easy to take the derivative. Therefore we'll let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x} \ dx$ and $v = x$. Thus

$$\int \ln x \ dx = x \ln x - \int x \cdot \frac{1}{x} \ dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

Since we have a definite integral, we can drop the "$+C$" and say

$$\int_a^1 \ln x = [x \ln x - x]_a^1 = ((1) \ln(1) - 1) - (a \ln a - a) = a - a \ln a - 1$$

Now we want to take this limit:

$$\lim_{a \to 0^+} a - a \ln a - 1 = \lim_{a \to 0^+} a - \lim_{a \to 0^+} a \ln a - \lim_{a \to 0^+} 1 = 0 - \lim_{a \to 0^+} a \ln a - 1$$

In order to calculate the last limit, we note that it is of the form $0 \cdot (-\infty)$. To solve this type of integral we first rewrite it as

$$\lim_{a \to 0^+} a \ln a = \lim_{a \to 0^+} \frac{\ln a}{\frac{1}{a}}$$

Now that it is in the form $\frac{-\infty}{\infty}$ we can apply L'Hôpital's Rule to get

$$\lim_{a \to 0^+} \frac{\ln a}{\frac{1}{a}} = \lim_{a \to 0^+} \frac{\frac{d}{da}[\ln a]}{\frac{d}{da}\left[\frac{1}{a}\right]}$$

$$= \lim_{a \to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}}$$

$$= \lim_{a \to 0^+} -\frac{a^2}{a}$$

$$= \lim_{a \to 0^+} a$$

$$= 0$$

Now that we have solved this limit we have

$$\int_0^1 \ln x \ dx = \lim_{a \to 0^+} \int_a^1 \ln x \ dx = \lim_{a \to 0^+} a - a \ln a - 1 = -1$$

Does this answer make sense? Let's look at the graph!

p.378 #49: The gamma funciton is defined for all $x > 0$ by the rule

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \ dt.$$

(a) Find $\Gamma(1)$ and $\Gamma(2)$.

(b) Integrate by parts with respect to t to show that, for positive n,

$$\Gamma(n+1) = n\Gamma(n).$$

(c) Find a simple expression for $\Gamma(n)$ for positive integers n.

(a) Plugging in $x = 1$ we need to solve

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t} \ dt = \int_0^\infty t^0 e^{-t} \ dt = \int_0^\infty e^{-t} \ dt.$$

This is an improper integral because the upper bound is ∞. To solve it, if it does converge, we'll use

$$\int_0^\infty e^{-t} \ dt = \lim_{b \to \infty} \int_0^b e^{-t} \ dt.$$

Now we can solve the integral. We'll skip the u-substitution to get:

$$\lim_{b \to \infty} \int_0^b e^{-t} \ dt = \lim_{b \to \infty} [-e^{-t}]_0^b$$

$$= \lim_{b \to \infty} -e^{-b} - (-e^0)$$

$$= \lim_{b \to \infty} 1 - \frac{1}{e^b}$$

As $b \to \infty$, e^b also grows without bound so $\frac{1}{e^b} \to 0$. Thus

$$\Gamma(1) = \lim_{b \to \infty} 1 - \frac{1}{e^b} = 1.$$

Now plugging in $x = 2$ we need to solve

$$\Gamma(2) = \int_0^\infty t^{2-1} e^{-t} \ dt = \int_0^\infty t e^{-t} \ dt.$$

Once again we're dealing with an improper integral. If it converges, we'll have

$$\int_0^\infty t e^{-t} \ dt = \lim_{b \to \infty} \int_0^b t e^{-t} \ dt.$$

This time the integral is a little harder to solve. We'll use integration by parts with $u = t$ and $dv = e^{-t} \ dt$. Then $du = dt$ and $v = -e^{-t} \ dt$ so

$$\lim_{b \to \infty} \int_0^b t e^{-t} = \lim_{b \to \infty} \left([-t e^{-t}]_0^b - \int -e^{-t} \ dt \right)$$

$$= \lim_{b \to \infty} \left((-b e^{-b}) - (-0 e^{-0}) + \int_0^b e^{-t} \ dt \right)$$

$$= \lim_{b \to \infty} \left(-b e^{-b} + \int_0^b e^{-t} \ dt \right)$$

$$= \lim_{b \to \infty} \left(-\frac{b}{e^b} + [-e^{-t}]_0^b \ dt \right)$$

(c) We need a function that satifies $f(1) = 1$ and $f(2) = 1$ and $f(n+1) = n \cdot f(n)$. We can calculate some values of f:

n	$f(n)$
1	1
2	1
3	$2 \cdot f(2) = 2 \cdot 1 = 2$
4	$3 \cdot f(3) = 3 \cdot 2 \cdot 1 = 6$
5	$4 \cdot f(4) = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

This looks almost like the factorial function, $f(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$. The difference is $2! = 2 \cdot 1 = 2$, not 1, so we have
$$\Gamma(n+1) = n!$$