p. 360, #45: Calculate

$$\int \frac{x-2}{x^2+x^4}\,dx.$$

First, note that

$$\int \frac{x-2}{x^2+x^4}\,dx = \int \frac{x-2}{x^2(1+x^2)}\,dx.$$

This gives us the following partial fraction decomposition

$$\begin{array}{rrcl} & \dfrac{x-2}{x^2(1+x^2)} & = & \dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{1+x^2} \\ \Rightarrow & x-2 & = & A(x)(1+x^2)+B(1+x^2)+(Cx+D)(x^2) \\ \Rightarrow & 0x^3+0x^2+1x-2 & = & Ax+Ax^3+B+Bx^2+Cx^3+Dx^2 \\ \Rightarrow & 0x^3+0x^2+1x-2 & = & (A+C)x^3+(B+D)x^2+Ax+B \end{array}$$

Because of the irreducible quadratic term and the repeated factor, we have to use the system of equations method, so

$$\begin{array}{rcl} 0 & = & A+C \\ 0 & = & B+D \\ 1 & = & A \\ -2 & = & B \end{array}$$

Solving gives $A=1, B=-2, C=-1, D=2$. Plugging these in gives

$$\int \frac{x-2}{x^2(1+x^2)}\,dx = \int \frac{1}{x}+\frac{-2}{x^2}+\frac{-x+2}{x^2+1}\,dx$$

We can break these up into individual integrals to get

$$\int \frac{1}{x}+\frac{-2}{x^2}+\frac{-x+2}{x^2+1}\,dx = \int \frac{1}{x}\,dx - 2\int \frac{1}{x^2}\,dx + \int \frac{-x+2}{x^2+1}\,dx$$

The first two are easy to solve:

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad\qquad \int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = -x^{-1}+C$$

and since we still have an integral we can drop the "$+C$" to get

$$\begin{aligned} \int \frac{1}{x}\,dx - 2\int \frac{1}{x^2}\,dx + \int \frac{-x+2}{x^2+1}\,dx &= \ln|x| - 2\left(-\frac{1}{x}\right) + \int \frac{-x+2}{x^2+1}\,dx \\ &= \ln|x| + \frac{2}{x} + \int \frac{-x+2}{x^2+1}\,dx \end{aligned}$$

Now we just need to solve

$$\int \frac{-x+2}{x^2+1}\,dx.$$

The naïve approach is to use a trig substitution. We'll do that first. Because the denominator is x^2+1, we will use $\tan\theta = x$. Then

$$\int \frac{-x+2}{x^2+1}\,dx = \int \frac{-\tan\theta+2}{\tan^2\theta+1}\,dx.$$

Note that we need to change our dx into a $d\theta$, so

$$\begin{aligned} \frac{d}{d\theta}[\tan\theta] &= \frac{d}{d\theta}[x] \\ \Rightarrow \quad \frac{1}{\cos^2\theta} &= \frac{dx}{d\theta} \\ \Rightarrow \quad \frac{1}{\cos^2\theta}\, d\theta &= dx \end{aligned}$$

and we get

$$\int \frac{-\tan\theta + 2}{\tan^2\theta + 1}\, dx = \int \frac{-\tan\theta + 2}{\tan^2\theta + 1} \cdot \frac{1}{\cos^2\theta}\, d\theta.$$

Remember that $\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}$ so

$$\begin{aligned} \int \frac{-\tan\theta + 2}{\tan^2\theta + 1} \cdot \frac{1}{\cos^2\theta}\, d\theta &= \int \frac{-\tan\theta + 2}{\left(\frac{\sin^2\theta}{\cos^2\theta} + 1\right)\cos^2\theta}\, d\theta \\ &= \int \frac{-\tan\theta + 2}{\sin^2\theta + \cos^2\theta}\, d\theta \end{aligned}$$

Now recall that $\sin^2\theta + \cos^2\theta = 1$, so

$$\int \frac{-\tan\theta + 2}{\sin^2\theta + \cos^2\theta}\, d\theta = \int -\tan\theta + 2\, d\theta.$$

We can break this integral up into two parts to get

$$\begin{aligned} \int -\tan\theta + 2\, d\theta &= -\int \tan\theta\, d\theta + 2\int d\theta \\ &= -\int \frac{\sin\theta}{\cos\theta}\, d\theta + 2\int d\theta. \end{aligned}$$

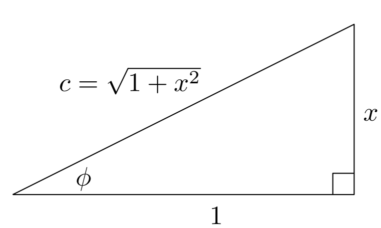
We can solve the first integral by using u substitution to with $u = \cos\theta$ (so $du = -\sin\theta\, d\theta$) to get

$$\begin{aligned} -\int \frac{\sin\theta}{\cos\theta}\, d\theta + 2\int d\theta &= -\int \frac{\sin\theta}{u} \cdot \frac{du}{-\sin\theta} + 2\int d\theta \\ &= \int \frac{1}{u}\, du + 2\int d\theta \\ &= \ln|u| + 2\theta + C \\ &= \ln|\cos\theta| + 2\theta + C \end{aligned}$$

Now we have to change our θs into xs:

$$\ln|\cos\theta| + 2\theta + C = \ln|\cos(\arctan(x))| + 2\arctan(x) + C.$$

We can simplify $\cos(\arctan(x))$ by setting $\varphi = \arctan(x)$ and drawing the triangle below:



$1^2 + x^2 = c^2$
so $c = \sqrt{1 + x^2}$

Then it is clear that $\cos(\arctan(x)) = \cos(\varphi) = \dfrac{1}{\sqrt{1+x^2}}$ and

$$\begin{aligned}
\ln|\cos(\arctan(x))| + 2\arctan(x) + C &= \ln\left|\frac{1}{\sqrt{1+x^2}}\right| + 2\arctan(x) + C \\
&= \ln|(1+x^2)^{-1/2}| + 2\arctan(x) + C \\
&= -\frac{1}{2}\ln|1+x^2| + 2\arctan(x) + C \\
&= -\frac{1}{2}\ln(1+x^2) + 2\arctan(x) + C \qquad \text{Because } 1+x^2>0 \text{ for all } x.
\end{aligned}$$

So we found

$$\int \frac{-x+2}{x^2+1}\,dx = -\frac{1}{2}\ln(1+x^2) + 2\arctan(x) + C$$

but it was a lot of work.

There is a more clever way to solve this intregral. Note that

$$\int \frac{-x+2}{x^2+1}\,dx = \int \frac{1}{x^2+1}(-x+2)\,dx = \int \frac{-x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx.$$

Breaking the integral into two integrals reveals that each is pretty easy to solve. For the first, we use a u substitution with $u = x^2+1$ (so $du = 2x\,dx$) to get

$$\begin{aligned}
\int \frac{-x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx &= \int \frac{-x}{u}\,\frac{du}{2x} + \int \frac{2}{x^2+1}\,dx \\
&= -\frac{1}{2}\int \frac{1}{u}\,du + \int \frac{2}{x^2+1}\,dx \\
&= -\frac{1}{2}\ln|u| + \int \frac{2}{x^2+1}\,dx \\
&= -\frac{1}{2}\ln|x^2+1| + \int \frac{2}{x^2+1}\,dx \\
&= -\frac{1}{2}\ln(x^2+1) + \int \frac{2}{x^2+1}\,dx \qquad \text{As } x^2+1>0 \text{ for all } x
\end{aligned}$$

For the second integral we notice that $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$ so we can pull out the two and get

$$\int \frac{-x+2}{x^2+1}\,dx = -\frac{1}{2}\ln(x^2+1) + 2\arctan(x) + C$$

which is just what we got with the trig substitution.

Finally, remember that all of this was just to solve the smaller integral in our main problem, so our final answer is

$$\int \frac{x-2}{x^2+x^4}\,dx = \ln|x| + \frac{2}{x} + \int \frac{-x+2}{x^2+1}\,dx = \ln|x| + \frac{2}{x} - \frac{1}{2}\ln(x^2+1) + 2\arctan(x) + C.$$