MATH 107-253 Recitation 12

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p. 383 #5: Does

$$\int_1^\infty \frac{x}{x^2+2x+4}\,dx$$

converge or diverge? Start by using "intuition analysis" to guess whether it converges or diverges and then use the comparison rule to show your guess is correct.

This integral is improper because the upper bound is infinity, so we want to look at the behavior of the function as $x \to \infty$. When x is large, the numerator clearly acts like x. In the denominator, x^2 gets larger much faster than $2x$ or 4, so the denominator acts like x^2. Thus our intution says this integral should act like $\frac{x}{x^2} = \frac{1}{x}$. We know $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges by the p-test, so we expect $\displaystyle\int_1^\infty \frac{x}{x^2+2x+4}\,dx$ to diverge as well.

In order to use the comparison test, we will need to find a function $f(x)$ such that

$$f(x) \leq \frac{x}{x^2+2x+4}$$

when $x \geq 1$ and $\int_1^\infty f(x)\,dx$ diverges. Unfortunately, using $\frac{1}{x}$ doesn't work because $x^2 < x^2+2x+4$ when $x \geq 1$. Instead, note that

$$\begin{aligned}
x^2 &\leq x^2 \\
2x &\leq 2x^2 \\
4 &\leq 4x^2 \\
x^2+2x+4 &\leq x^2+2x^2+4x^2 \\
\frac{1}{x^2+2x+4} &\geq \frac{1}{x^2+2x^2+4x^2} \\
\frac{x}{x^2+2x+4} &\geq \frac{x}{x^2+2x^2+4x^2} = \frac{1}{7x}
\end{aligned}$$

where in each case we use $x \geq 1$. Thus for our $f(x)$ we use $\frac{1}{7x}$. We know that

$$\int_1^\infty \frac{1}{7x}\,dx = \frac{1}{7}\int_1^\infty \frac{1}{x}\,dx \quad \text{which diverges by } p\text{-test.}$$

Now we can finally say:

Because $\frac{1}{7x} \leq \frac{x}{x^2+2x+4}$ when $x \geq 1$ and $\displaystyle\int_1^\infty \frac{1}{7x}\,dx$ diverges by the p-test, we know $\int_1^\infty \frac{x}{x^2+2x+4}\,dx$ diverges by the comparison test.

p. 384 #20: Does

$$\int_1^\infty \frac{1}{\sqrt{\theta^3+\theta}}\ d\theta$$

converge or diverge? Start by using "intuition analysis" to guess whether it converges or diverges and then use the comparison rule to show your guess is correct.

Note: I copied the problem incorrectly. #20 in the book asks about $\displaystyle\int_0^1 \frac{1}{\sqrt{\theta^3+\theta}}\ d\theta$. We will solve both problems.

This integral is improper because the upper bound is infinity, so we want to look at the behavior of the function as $\theta \to \infty$. When θ is large, the numerator clearly acts like 1. In the denominator, θ^3 gets larger much faster than θ, so the denominator acts like $\sqrt{\theta^3}$. Thus our intution says this integral should act like $\frac{1}{\theta^{3/2}}$. We know $\displaystyle\int_1^\infty \frac{1}{\theta^{3/2}}\ d\theta$ converges by the p-test, so we expect $\displaystyle\int_1^\infty \frac{1}{\sqrt{\theta^3+\theta}}\ d\theta$ to converge as well.

In order to use the comparison test, we will need to find a function $g(\theta)$ such that

$$g(\theta) \geq \frac{1}{\sqrt{\theta^3+\theta}}$$

when $\theta \geq 1$ and $\int_1^\infty g(\theta)\ d\theta$ converges. This time we **can** just use $\dfrac{1}{\theta^{3/2}}$ because

$$\begin{aligned}\theta^3+\theta &\geq \theta^3\\ \sqrt{\theta^3+\theta} &\geq \sqrt{\theta^3}\\ \frac{1}{\sqrt{\theta^3+\theta}} &\leq \frac{1}{\sqrt{\theta^3}} = \frac{1}{\theta^{3/2}}\end{aligned}$$

We know that

$$\int_1^\infty \frac{1}{\theta^{3/2}}\ d\theta \text{ converges by } p\text{-test.}$$

Now we can finally say:

Because $\frac{1}{\theta^{3/2}} \geq \frac{1}{\sqrt{\theta^3+\theta}}$ when $\theta \geq 1$ and $\displaystyle\int_1^\infty \frac{1}{\theta^{3/2}}\ d\theta$ converges by the p-test, we know $\displaystyle\int_1^\infty \frac{1}{\sqrt{\theta^3+\theta}}\ d\theta$ converges by the comparison test.

Now we solve the question the book actually asked about: Does

$$\int_0^1 \frac{1}{\sqrt{\theta^3+\theta}}\ d\theta$$

converge or diverge?

This integral is improper because the function is undefined at $\theta = 0$, so we want to look at the behavior of the function as $\theta \to 0$. When θ is small, the numerator clearly acts like 1. In the denominator, θ^3 gets smaller much faster than θ, so the denominator acts like $\sqrt{\theta}$. Thus our intution says this integral should act like $\frac{1}{\theta^{1/2}}$. We know $\displaystyle\int_0^1 \frac{1}{\theta^{1/2}}\ d\theta$ converges by the p-test, so we expect $\displaystyle\int_0^1 \frac{1}{\sqrt{\theta^3+\theta}}\ d\theta$ to converge as well.

In order to use the comparison test, we will need to find a function $g(\theta)$ such that

$$g(\theta) \geq \frac{1}{\sqrt{\theta^3+\theta}}$$

when $0 \leq \theta \leq 1$ and $\int_0^1 g(\theta)\, d\theta$ converges. Once again we **can** just use $\frac{1}{\theta^{1/2}}$ because

$$\begin{aligned} \theta^3+\theta &\geq \theta \\ \sqrt{\theta^3+\theta} &\geq \sqrt{\theta} \\ \frac{1}{\sqrt{\theta^3+\theta}} &\leq \frac{1}{\sqrt{\theta}} = \frac{1}{\theta^{1/2}} \end{aligned}$$

We know that

$$\int_0^1 \frac{1}{\theta^{1/2}}\, d\theta \text{ converges by } p\text{-test.}$$

Now we can finally say:

Because $\frac{1}{\theta^{1/2}} \geq \frac{1}{\sqrt{\theta^3+\theta}}$ when $0 \leq \theta \leq 1$ and $\displaystyle\int_0^1 \frac{1}{\theta^{1/2}}\, d\theta$ converges by the p-test, we know $\displaystyle\int_0^1 \frac{1}{\sqrt{\theta^3+\theta}}\, d\theta$ converges by the comparison test.

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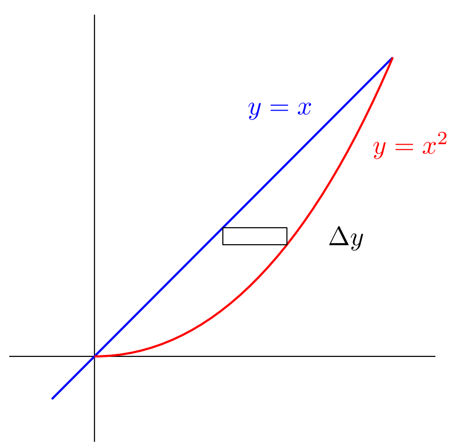
p. 389 #7: Write a Riemann sum and then a definite integral representing the area between the curves $y = x$ and $y = \sqrt{x}$ using horizontal strips of height Δy.

The first thing we should do is sketch a picture of what the question is asking:



Next we need to ask when these curves intersect. That happens when

$$\begin{aligned} x &= x^2 \\ x - x^2 &= 0 \\ x(1-x) &= 0 \end{aligned}$$

This happens when $x = 0$ or $x = 1$. Using either of the equations, we get the points of intersection as $(0,0)$ and $(1,1)$.

Now usually we could choose whether we wanted to use horizontal strips of height Δy or vertical strips of width Δx, but the problem specifies that we need to do the former.

How wide are our horizontal strips? The width goes all the over to the outside curve, $y = x^2$, but we need to subtract off the inside curve, $y = x$. Thus the width is $x^2 - x$ and the area of the rectangle is $(x^2 - x)\Delta y$. But this doesn't work because we have mixed xs and ys. We need to turn the xs in our width into ys. To do this we use the inverse functions $\sqrt{y} = x$ and $y = x$ to get the area of the rectangle as $(\sqrt{y} - y)\Delta y$.

Now we can add up all of the rectangles to get the Riemann sum

$$\sum_{i=1}^{n} (\sqrt{y_i} - y_i)\Delta y$$

Now taking the limit as $\Delta y \to 0$ (or, equivalently, $n \to \infty$), we get the integral

$$\int_0^1 \sqrt{y} - y \, dy$$

where the bounds are 0 to 1 becuase the y-coordinates of the intersection points are 0 and 1.

The problem doesn't ask for it, but we can solve this integral easily to get $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.

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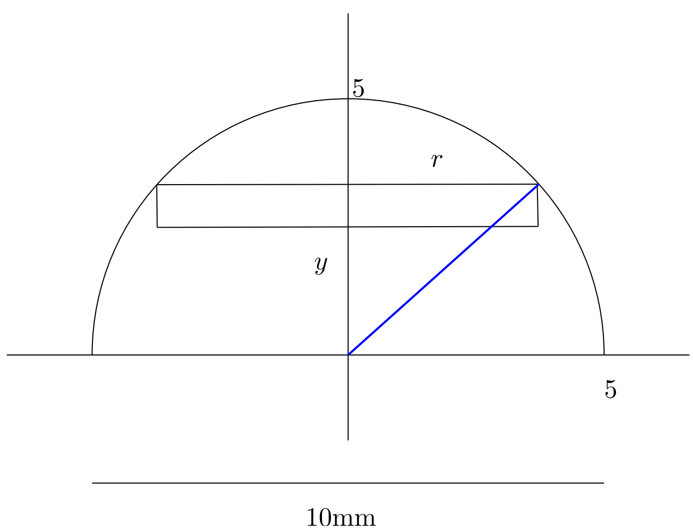
p. 389 #13: Write a Riemann sum and then a definite integral representing the volume of a hemisphere with diameter 10mm using horizontal strips of height Δy.

Once again we **should** start by drawing a picture. However, it is very difficult to do so digitally, so these solutions omit the picture. Sorry.

If we could look at the picture, we would see that our approximation is a disk with height Δy and some radius r. The volume of the disk is thus $(\pi r^2)\Delta y$ but we'll need to find a formula for r in terms of y. To do that, we "squash" the picture down to two dimensions and see this:



In order to find r, we draw the blue line to create a right triangle. Then we can use the Pythagorean Theorem to get

$$y^2 + r^2 = 5^2 \text{ or } r = \sqrt{25 - y^2}$$

Therefore the Riemann sum for our disks is

$$\sum_{i=1}^{n}(\pi(\sqrt{25-y_i^2})^2)\Delta y$$

and taking the limit as $\Delta y \to 0$ gives us

$$\pi \int_0^5 25 - y^2 \; dy$$

where we get the bounds from the picture: the height of the disks range from $y = 0$ to $y = 5$.

Evaluating the integral gives us $\frac{2}{3}\pi 5^3 \text{mm}^3$ which is exactly what we expect from geometry.

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p. 389 #21: Consider

$$\int_0^6 \pi \left(3 - \frac{y}{2}\right)^2 \; dy.$$

Is this the formula for the volume of a hemisphere or a cone? If a hemisphere, give its radius; if a cone, give its height. Sketch your solid and label all relevant quantities.

The first thing we should do is turn the integral back into a Riemann sum:

$$\sum_{i=1}^{n} \pi \left(3 - \frac{y_i}{2}\right)^2 \Delta y.$$

This makes it easy to see that our estimation is the sum of disks with radius $3 - \frac{y}{2}$. This radius changes linearly with y, so we expect this integral to describe the volumne of a cone, not a hemisphere.

Looking at the integral, the range of y is 0 to 6, so our cone must have height 6. When $y = 0$, our radius is three, so the radius of the cone must be three. Once again, it is difficult to draw three-dimensional pictures on the computer, so no picture is included.