

## MATH 107-253 Recitation 17

JD Nir

Avery 230 • Office Hours: W 4-5 R 11-12

jnir@huskers.unl.edu

www.math.unl.edu/~jnir2/107-253.html

Tuesday, October 27, 2015

---

**p. 422 #12:** The density of oil in a circular oil slick on the surface of the ocean at a distance  $r$  meters from the center of the slick is given by  $\delta(r) = 50/(1+r)$  kg/m<sup>2</sup>.

- (a) If the slick extends from  $r = 0$  to  $r = 10,000$  m, find a Riemann sum approximating the total mass of oil in the slick.
- (b) Find the exact value of the mass of oil in the slick by turning your sum into an integral and evaluating it.
- (c) Within what distance  $r$  is half the oil of the slick contained?

---

(a) If we look at the disk at radius  $r$  from the center of the spill, with width  $\Delta r$ , the area of the disk is approximately  $2\pi r \Delta r$  and has constant density  $\frac{50}{1+r}$  so has weight  $100\pi \frac{r}{1+r} \Delta r = 100\pi(1 - \frac{1}{1+r})\Delta r$ . If we have  $n$  such rings, we can approximate the total area of the spill as

$$\sum_{i=1}^n 100\pi(1 - \frac{1}{1+r})\Delta r.$$

(b) Taking the limit as  $\Delta r \rightarrow 0$ , and noting that the radius changes from  $r = 0$  to  $r = 10,000$ , we get the total mass of the oil is

$$\int_0^{10000} 100\pi(1 - \frac{1}{1+r}) dr = 100\pi[r - \ln(1+r)]_0^{10000} = 100\pi(10000 - \ln(10001)).$$

(c) Let  $R$  be the distance such that half of the oil is contained in radius  $R$ . Then

$$\int_0^R 100\pi(1 - \frac{1}{1+r}) dr = \frac{1}{2} \int_0^{10000} 100\pi(1 - \frac{1}{1+r}) dr = \frac{1}{2}(100\pi(10000 - \ln(10001))) = 50\pi(10000 - \ln(10001)).$$

We can evaluate the integral to get

$$\int_0^R 100\pi(1 - \frac{1}{1+r}) dr = 100\pi[r - \ln(1+r)]_0^R = 100\pi(R - \ln(1+R)).$$

Thus we seek  $R$  such that

$$\begin{aligned} 100\pi(R - \ln(1+R)) &= 50\pi(10000 - \ln(10001)) \implies 2R - \ln(1+R)^2 = 10000 - \ln(10001) \\ &\implies 2R - \ln(1+R)^2 - 10000 + \ln(10001) = 0. \end{aligned}$$

Using a computer, we can find the zeroes of that function to get  $R \approx 5003.9$

**p. 431 #7: An anchor weighing 100 lb in water is attached to a chain weighing 3 lb/ft in water. Find the work done to haul the anchor and chain to the surface of the water from a depth of 25 ft.**

---

Break up the chain into segments of height  $\Delta h$ . If we have already removed  $h$  feet of chain from the water, then the rest of the chain weighs  $3(25 - h)$  pounds. Of course, the anchor has constant weight of 100 pounds. We can approximate the total work by calculating the work done to pull each segment of height  $\Delta h$  by

$$\text{Work} = \text{Force} \cdot \text{Distance} = \sum_{i=1}^n (3(25 - h) + 100) \Delta h.$$

By taking  $\Delta h \rightarrow 0$ , noting that  $h$  changes from 0 to 25, we get the total work is

$$\int_0^{25} 3(25 - h) + 100 \, dh = \int_0^{25} 175 - 3h \, dh = \left[ 175h - \frac{3}{2}h^2 \right]_0^{25} = \frac{6875}{2} \text{ft} \cdot \text{lbs}.$$

---

**p. 431 #15: Water in a cylinder of height 10 ft and radius 4 ft is to be pumped out. Using the fact that the pressure of water at depth  $h$  is  $62.4h$  lb/ft<sup>2</sup>, find the work required if**

- (a) The tank is full of water and the water is to be pumped over the top of the tank.
  - (b) The tank is full of water and the water must be pumped to a height of 5 ft above the top of the tank.
  - (c) The depth of water in the tank is 8 ft and the water must be pumped over the top of the tank.
- 

(a) Consider a disk of height  $\Delta h$  at height  $h$  from the top of the tank. The disk contains  $\pi r^2 \Delta h$  water with  $r = 4$ . The work to bring it to the top of the tank is approximately  $62.4h$ . Thus our Riemann sum is

$$\sum_{i=1}^n 16\pi(62.4h) \Delta h.$$

Taking  $\Delta h \rightarrow 0$  and noting we start pumping our water from height 0 and end at height 10, we get

$$\text{Work} = \int_0^{10} 16\pi \cdot 62.4h \, dh = 16\pi \cdot 62.4 \left[ \frac{1}{2}h^2 \right]_0^{10} = 49920\pi \text{ft} \cdot \text{lbs}.$$

(b) We can reuse a lot of our work. The only thing that changes is instead of doing  $62.4h$  work to our disk of water, we now need to do  $62.4(5 + h)$  work. This gives us the new integral

$$\text{Work} = \int_0^{10} 16\pi \cdot 62.4(5 + h) \, dh = 16\pi \cdot 62.4 \left[ 5h + \frac{1}{2}h^2 \right]_0^{10} = 99840\pi \text{ft} \cdot \text{lbs}.$$

(c) Once again we can reuse a lot of the work from part (a). In this case, what changes are the bound of our integral. Instead of starting at  $h = 0$ , we start at  $h = 2$  because the water starts 2ft below the top of the tank. Thus

$$\text{Work} = \int_2^{10} 16\pi \cdot 62.4h \, dh = 16\pi \cdot 62.4 \left[ \frac{1}{2}h^2 \right]_2^{10} = 47932.2\pi \text{ft} \cdot \text{lbs}.$$

Note that there are a few different ways of approaching this problem. You could instead define  $h$  as the height from the bottom of the tank, in which case the work to bring a disk of water to the top is  $62.4(10 - h)$  or  $62.4(15 - h)$  to pump the water 5 ft above the tank. I chose to define  $h$  as the distance from the top of the tank because it made the calculations easier.

---

**4: A bucket of water of mass 20 kg is pulled at constant velocity up to a platform 50 meters above the ground. This takes 20 minutes, during which time 4 kg of water drips out at a steady rate through a hole in the bottom. Find the work needed to raise the bucket to the platform. (Use  $g = 9.8 \frac{m}{s^2}$ .)**

---

To analyze this problem, consider the work it will take to raise the bucket  $\Delta h$  when it is at height  $h$  from the ground. To do this, we will need to know the force it takes to overcome gravity, and to know that we will need to know the weight of the bucket at height  $h$ . We will assume the bucket has constant weight while it is being raised  $\Delta h$ .

Note that 20 minutes is 1200 seconds, so the rate that water is dripping out is  $\frac{4}{1200} = \frac{1}{300} \frac{\text{kg}}{\text{s}}$ . Since the velocity used to pull up the bucket is constant, it must be  $\frac{50}{1200} = \frac{1}{24} \frac{\text{m}}{\text{s}}$ . Thus at height  $h$ , we have been pulling for  $24h$  seconds and have lost  $\frac{24h}{300} = \frac{2}{25}h$  kg of water, meaning the weight is  $20 - \frac{2}{25}h$  kg. Thus the force needed to overcome gravity is  $9.8(20 - \frac{2}{25}h)\Delta h$ .

Adding up all of our approximations gives us the Riemann sum

$$\sum_{i=1}^n 9.8(20 - \frac{2}{25}h_i)\Delta h.$$

Taking the limit as  $\Delta h \rightarrow 0$ , and noting that  $h$  starts at 0 and goes to 50, we get

$$\text{Work} = \int_0^{50} 9.8(20 - \frac{2}{25}h) \, dh = 9.8 \left[ 20h - \frac{1}{25}h^2 \right]_0^{50} = 8820J.$$