# MATH 107-153 Recitation 10

JD Nir
Avery 230 • Office Hours: W 4-5 R 11-2
jnir@huskers.unl.edu
www.math.unl.edu/∼jnir2/107-153.html
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**1: Integrate each of the following without using an integration table.**

$$\text{(a)} \int \frac{1}{\sqrt{1-u^2}}\,du \quad \text{(b)} \int \frac{1}{1-u^2}\,du \quad \text{(c)} \int \frac{u}{\sqrt{1-u^2}}\,du \quad \text{(d)} \int \frac{u}{1-u^2}\,du.$$

**Simplify your answer as much as possible.**

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(a) You may recognize this integral as the derivative of $\arcsin(x)$. While we'll hope to get this answer, let's use a trig substitution to find it.

Because we have the form $a^2 - x^2$ (where $a = 1$ and $x = u$) we should use

$$\sin\theta = \frac{u}{1} = u.$$

Substituting gives

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \int \frac{1}{\sqrt{1-(\sin\theta)^2}}\,du.$$

Notice that we have mixed variables: our integral is in terms of $\theta$ but our differential is in terms of $du$. So we'll have to find an equation for $du$ in terms of $d\theta$.

$$\frac{d}{du}[\sin\theta] = \frac{d}{du}[u] \implies \cos\theta \cdot \frac{d\theta}{du} = 1 \implies \cos\theta\;d\theta = du$$

So now we can substitute and get:

$$\int \frac{1}{\sqrt{1-(\sin\theta)^2}}\,du = \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta.$$

Now we use the trig identity

$$\sin^2\theta + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \sin^2\theta$$

to get

$$\begin{aligned} \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta &= \int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta \\ &= \int \frac{\cos\theta}{\cos\theta}\,d\theta \\ &= \int 1\;d\theta \\ &= \theta + C \end{aligned}$$

Now we need to change our answer from being in terms of $\theta$ to being in terms of $u$. We have

$$\sin\theta = u \implies \arcsin(\sin\theta) = \arcsin u \implies \theta = \arcsin u$$

so

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u) + C.$$

(b) Even though this integral looks very similar to the last one, using the substitution $u = \sin\theta$ leads us to the integral

$$\int \frac{1}{\cos\theta}\, d\theta$$

which is very hard to solve. (Go ahead and confirm this is true by trying the substitution!)

Instead, this is a partial fraction decomposition problem! First we have to factor the denominator to get

$$\int \frac{1}{1-u^2}\, du = \int \frac{1}{(1+u)(1-u)}\, du.$$

Now we can do the partial fraction decomposition:

$$\begin{aligned} \frac{1}{(1+u)(1-u)} &= \frac{A}{1+u} + \frac{B}{1-u} \\ \implies \quad 1 &= A(1-u) + B(1+u) \end{aligned}$$

Now there are two ways we can proceed. The first is to note that $1 = 0u + 1$ is a polynomial in terms of $u$ and so is $A(1-u) + B(1+u) = (B-A)u + (A+B)$. Because these two polynomials are equal, the coefficients of each term must be equal. This gives us a system of equations with two equations and two unknowns:

$$\begin{aligned} 0 &= (B-A) \\ 1 &= (A+B) \end{aligned}$$

We can solve this system to find $A = \frac{1}{2}$ and $B = \frac{1}{2}$.

Alternatively, we know $1 = A(1-u) + B(1+u)$ has to hold for all values of $u$. In particular, let $u = 1$. Then

$$1 = A(1-1) + B(1+1) \implies 1 = 2B \implies B = \frac{1}{2}.$$

Also, let $u = -1$ and

$$1 = A(1-(-1)) + B(1+(-1)) \implies 1 = 2A \implies A = \frac{1}{2}.$$

As expected, both methods give us the same answer.

Now we can use our partial fraction decompostion to write

$$\begin{aligned} \int \frac{1}{(1+u)(1-u)}\, du &= \int \frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u}\, du \\ &= \frac{1}{2}\int \frac{1}{1+u}\, du + \frac{1}{2}\int \frac{1}{1-u}\, du \end{aligned}$$

While these are almost of the form $\int \frac{1}{x}\, dx = \ln|x|$, they aren't quite that simple. We may be tempted to skip the substitution step, but that would be dangerous in this case for reasons we'll see in a moment. Because our variable is $u$, we'll do a $w$-substitution.

For the first substitution we get $w = 1 + u$. Then $dw = du$ and

$$\frac{1}{2}\int \frac{1}{1+u}\, du = \frac{1}{2}\int \frac{1}{w}\, dw = \frac{1}{2}\ln|w| + C_1 = \frac{1}{2}\ln|1+u| + C_1.$$

For the second substitution we use $w = 1 - u$. Then $dw = -du$. Note the minus sign! If we'd skipped the substitution step, it's possible we would have missed it!

$$\frac{1}{2}\int \frac{1}{1-u}\, du = \frac{1}{2}\int \frac{1}{w}(-dw) = -\frac{1}{2}\int \frac{1}{w}\, dw = -\frac{1}{2}\ln|w| + C_2 = -\frac{1}{2}\ln|1-u| + C_2$$

Now we can combine those two equations. We can also combine the constants into one constant.

$$\frac{1}{2}\int \frac{1}{1+u}\ du + \frac{1}{2}\int \frac{1}{1-u}\ du = \frac{1}{2}\ln|1+u| - \frac{1}{2}\ln|1-u| + C$$

Because the question asks us to simplify as much as possible, we should use some of the laws of logs:

$$\begin{aligned}
\frac{1}{2}\ln|1+u| - \frac{1}{2}\ln|1-u| + C &= \frac{1}{2}(\ln|1+u| - \ln|1-u|) + C \\
&= \frac{1}{2}\left(\ln\left|\frac{1+u}{1-u}\right|\right) + C \\
&= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C \\
&= \ln\left|\frac{1+u}{1-u}\right|^{1/2} + C \\
&= \ln\sqrt{\left|\frac{1+u}{1-u}\right|} + C
\end{aligned}$$

This gives us a final answer of

$$\int \frac{1}{1-u^2}\ du = \ln\sqrt{\left|\frac{1+u}{1-u}\right|} + C.$$

This is a pretty complicated answer. Let's make sure it's right by taking the derivative.

$$\begin{aligned}
\frac{d}{du}\left[\ln\sqrt{\left|\frac{1+u}{1-u}\right|} + C\right] &= \frac{d}{du}\left[\ln\sqrt{\left|\frac{1+u}{1-u}\right|}\right] + \frac{d}{du}[C] \\
&= \frac{1}{\sqrt{\left|\frac{1+u}{1-u}\right|}} \cdot \frac{d}{du}\left[\sqrt{\left|\frac{1+u}{1-u}\right|}\right] + 0 \\
&= \frac{1}{\frac{\sqrt{|1+u|}}{\sqrt{|1-u|}}} \cdot \frac{1}{2\sqrt{\left|\frac{1+u}{1-u}\right|}} \cdot \frac{d}{du}\left[\left|\frac{1+u}{1-u}\right|\right] \\
&= \frac{\sqrt{|1-u|}}{\sqrt{|1+u|}} \cdot \frac{1}{2\frac{\sqrt{|1+u|}}{\sqrt{|1-u|}}} \cdot \frac{(1-u)[1]-[-1](1+u)}{(1-u)^2} \\
&= \frac{\sqrt{|1-u|}}{\sqrt{|1+u|}} \cdot \frac{\sqrt{|1-u|}}{2\sqrt{|1+u|}} \cdot \frac{1-u+1+u}{(1-u)^2} \\
&= \frac{\sqrt{|1-u|}^2}{2\sqrt{|1+u|}^2} \cdot \frac{2}{(1-u)^2} \\
&= \frac{|1-u|}{|1+u|(1-u)^2} \\
&= \frac{1}{(1+u)(1-u)} \\
&= \frac{1}{1-u^2}
\end{aligned}$$

Just what we wanted!

(c) This is another trig substitution. Since we have $a^2 - x^2$ in the denominator (with $a = 1$ and $x = u$ again) we'll use

$$\sin\theta = \frac{u}{1} = u$$

to get

$$\int \frac{u}{\sqrt{1-u^2}}\,du = \int \frac{u}{\sqrt{1-(\sin\theta)^2}}\,du.$$

We can get rid of the $u$ in the numerator by writing

$$\int \frac{u}{\sqrt{1-(\sin\theta)^2}}\,du = \int \frac{\sin\theta}{\sqrt{1-(\sin\theta)^2}}\,du.$$

Notice that we have mixed variables: our integral is in terms of $\theta$ but our differential is in terms of $du$. So we'll have to find an equation for $du$ in terms of $d\theta$.

$$\frac{d}{du}[\sin\theta] = \frac{d}{du}[u] \implies \cos\theta \cdot \frac{d\theta}{du} = 1 \implies \cos\theta\ d\theta = du$$

So now we can write

$$\int \frac{\sin\theta}{\sqrt{1-(\sin\theta)^2}}\,du = \int \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\ d\theta.$$

We can use the trig identity $\cos^2\theta = 1 - \sin^2\theta$ again to get

$$\begin{aligned}\int \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\ d\theta &= \int \frac{\sin\theta\cos\theta}{\sqrt{\cos^2\theta}}\ d\theta \\ &= \int \frac{\sin\theta\cos\theta}{\cos\theta}\ d\theta \\ &= \int \sin\theta\ d\theta \\ &= -\cos\theta + C\end{aligned}$$

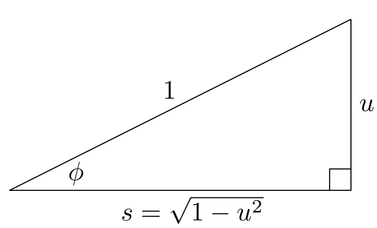
Now we solve for $\theta$:

$$\sin\theta = u \implies \arcsin(\sin\theta) = \arcsin(u) \implies \theta = \arcsin(u)$$

and get

$$-\cos\theta + C = -\cos(\arcsin(u)) + C.$$

We can simplify this answer further. Remember that the definition of $\arcsin(u)$ is "the angle whose sine is $u$." We can look at what that angle must look like. Lets call that angle $\phi$. Remember that the sine of an angle is the ratio of the side of the triangle opposite the angle over the hypotenues of the triangle. Then we can use the Pythagorean Theorem to fill in the last side of the triangle.




$$\begin{aligned} u^2 + s^2 &= 1^2 \\ s^2 &= 1 - u^2 \\ s &= \sqrt{1-u^2}\end{aligned}$$

Now let's return to simplifying $\cos(\arcsin(u))$. Since we set $\arcsin(u) = \phi$, we can look at the triangle to find

$$\cos(\arcsin(u)) = \cos(\phi) = \sqrt{1-u^2}$$

giving us a final answer of

$$\int \frac{u}{\sqrt{1-u^2}}\,du = \sqrt{1-u^2} + C.$$

Pretty amazing! Even though we used a trig substitution, the final answer has nothing to do with trig functions. Just to make sure we got it right, let's take the derivative to check.

$$\begin{aligned}
\frac{d}{du}\left[\sqrt{1-u^2} + C\right] &= \frac{d}{du}\left[\sqrt{1-u^2}\right] + \frac{d}{du}[C] \\
&= \frac{d}{du}[(1-u^2)^{1/2}] + 0 \\
&= -\frac{1}{2} \cdot (1-u^2)^{-1/2} \cdot \frac{d}{du}[1-u^2] \\
&= -\frac{1}{2(1-u^2)^{1/2}} \cdot (-2u) \\
&= -\frac{-2u}{2\sqrt{1-u^2}} \\
&= \frac{u}{\sqrt{1-u^2}}
\end{aligned}$$

which is exactly what we were expecting.

(d) Finally an easy problem! Don't be tricked by the other three; we can just use substituton to solve this one. Once again we'll use $w$ instead of $u$ because our variable is $u$. Let $w = 1 - u^2$. Then $\frac{dw}{du} = -2u$ and $du = \frac{dw}{-2u}$ so

$$\begin{aligned}
\int \frac{u}{1-u^2}\,du &= \int \frac{u}{w}\frac{dw}{-2u} \\
&= -\frac{1}{2}\int \frac{1}{w}\,dw \\
&= -\frac{1}{2}\ln|w| + C \\
&= -\frac{1}{2}\ln|1-u^2| + C
\end{aligned}$$

**p. 378 #7: Find**

$$\int_0^1 \ln x \; dx$$

**or explain why it does not converge.**

Note that ln(0) is not defined so we need to treat this as an indefinite integral. So if this integral does converge we will have

$$\int_0^1 \ln(x) \; dx = \lim_{a\to 0^+} \int_a^1 \ln(x) \; dx.$$

Now let's look at the integral. We don't have an integration rule that solves this integral, but we can solve it with integration by parts.

While we don't know how to take the integral of $\ln x$, it is easy to take the derivative. Therefore we'll let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}\, dx$ and $v = x$. Thus

$$\begin{aligned} \int \ln x \; dx &= x \ln x - \int x \cdot \frac{1}{x} \; dx \\ &= x \ln x - \int dx \\ &= x \ln x - x + C \end{aligned}$$

Since we have a definite integral, we can drop the "$+C$" and say

$$\int_a^1 \ln x = [x \ln x - x]_a^1 = ((1)\ln(1) - 1) - (a \ln a - a) = a - a \ln a - 1$$

Now we want to take this limit:

$$\lim_{a\to 0^+} a - a\ln a - 1 = \lim_{a\to 0^+} a - \lim_{a\to 0^+} a \ln a - \lim_{a\to 0^+} 1 = 0 - \lim_{a\to 0^+} a \ln a - 1$$

In order to calculate the last limit, we note that it is of the form $0 \cdot (-\infty)$. To solve this type of integral we first rewrite it as

$$\lim_{a\to 0^+} a \ln a = \lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a}}$$

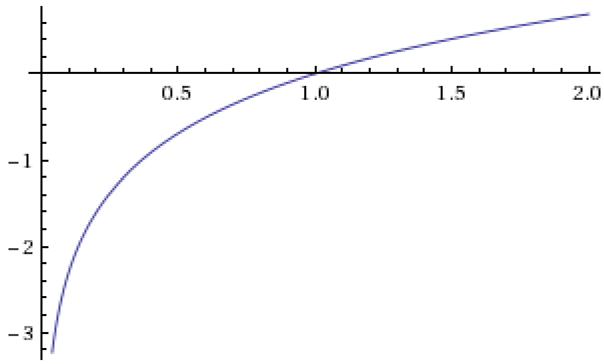
Now that it is in the form $\frac{-\infty}{\infty}$ we can apply L'Hôpital's Rule to get

$$\begin{aligned} \lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a}} &= \lim_{a\to 0^+} \frac{\frac{d}{da}[\ln a]}{\frac{d}{da}\left[\frac{1}{a}\right]} \\ &= \lim_{a\to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}} \\ &= \lim_{a\to 0^+} -\frac{a^2}{a} \\ &= \lim_{a\to 0^+} a \\ &= 0 \end{aligned}$$

Now that we have solved this limit we have

$$\int_0^1 \ln x \; dx = \lim_{a\to 0^+} \int_a^1 \ln x \; dx = \lim_{a\to 0^+} a - a \ln a - 1 = -1$$

Does this answer make sense? Let's look at the graph!




The area of the graph between 0 and 1 is below the $x$-axis, so it makes sense that the integral is negative. Also, the area seems pretty reasonable as the area under the graph is almost the same as the triangle with base 1 and height $-2$ which has area $\frac{1}{2}(1)(-2) = -1$.

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**p.378 #49: The gamma funciton is defined for all $x > 0$ by the rule**

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\ dt.$$

**(a) Find $\Gamma(1)$ and $\Gamma(2)$.**

**(b) Integrate by parts with respect to $t$ to show that, for positive $n$,**

$$\Gamma(n+1) = n\Gamma(n).$$

**(c) Find a simple expression for $\Gamma(n)$ for positive integers $n$.**

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(a) Plugging in $x = 1$ we need to solve

$$\Gamma(1) = \int_0^\infty t^{1-1}e^{-t}\ dt = \int_0^\infty t^0 e^{-t}\ dt = \int_0^\infty e^{-t}\ dt.$$

This is an improper integral because the upper bound is $\infty$. To solve it, if it does converge, we'll use

$$\int_0^\infty e^{-t}\ dt = \lim_{b\to\infty}\int_0^b e^{-t}\ dt.$$

Now we can solve the integral. We'll skip the $u$-substitution to get:

$$\begin{aligned}\lim_{b\to\infty}\int_0^b e^{-t}\ dt &= \lim_{b\to\infty}[-e^{-t}]_0^b \\ &= \lim_{b\to\infty} -e^{-b} - (-e^0) \\ &= \lim_{b\to\infty} 1 - \frac{1}{e^b}\end{aligned}$$

As $b \to \infty$, $e^b$ also grows without bound so $\frac{1}{e^b} \to 0$. Thus

$$\Gamma(1) = \lim_{b\to\infty} 1 - \frac{1}{e^b} = 1.$$

Now plugging in $x = 2$ we need to solve

$$\Gamma(2) = \int_0^\infty t^{2-1}e^{-t}\ dt = \int_0^\infty te^{-t}\ dt.$$

Once again we're dealing with an improper integral. If it converges, we'll have

$$\int_0^\infty te^{-t}\ dt = \lim_{b\to\infty}\int_0^b te^{-t}\ dt.$$

This time the integral is a little harder to solve. We'll use integration by parts with $u = t$ and

$dv = e^{-t}\ dt$. Then $du = dt$ and $v = -e^{-t}\ dt$ so

$$\begin{aligned}
\lim_{b\to\infty}\int_0^b te^{-t} &= \lim_{b\to\infty}\left([-te^{-t}]_0^b - \int -e^{-t}\ dt\right) \\
&= \lim_{b\to\infty}\left((-be^{-b}) - (-0e^{-0}) + \int_0^b e^{-t}\ dt\right) \\
&= \lim_{b\to\infty}\left(-be^{-b} + \int_0^b e^{-t}\ dt\right) \\
&= \lim_{b\to\infty}\left(-\frac{b}{e^b} + [-e^{-t}]_0^b\ dt\right) \\
&= \lim_{b\to\infty}\left(-\frac{b}{e^b} + -e^{-b} - (-e^0)\ dt\right) \\
&= \lim_{b\to\infty}\left(-\frac{b}{e^b} - \frac{1}{e^b} + 1\ dt\right)
\end{aligned}$$

As we saw earlier, $\lim\limits_{b\to\infty}\frac{1}{e^b} = 0$. To solve $\lim\limits_{b\to\infty}\frac{b}{e^b}$ we could use L'Hôpital's Rule but instead we note that $e^b$ *dominates* $b$; that is, it grows much faster than $b$. This will be very important when we study other limits later in the course. For now, we accept $\lim\limits_{b\to\infty}\frac{b}{e^b} = 0$. Then

$$\Gamma(2) = \lim_{b\to\infty} -\frac{b}{e^b} - \frac{1}{e^b} + 1 = 1.$$

(b) Just like other improper integrals, we write

$$\Gamma(n+1) = \int_0^\infty t^{n+1-1}e^{-t}\ dt = \lim_{b\to\infty}\int_0^b t^n e^{-t}\ dt$$

As the problem suggests, we'll use integration by parts. As we've seen in these cases, it's usually best to let $u = t^n$ and $dv = e^{-t}\ dt$. Then $du = nt^{n-1}dt$ and $v = -e^{-t}$. Keep in mind that we are dealing with an improper integral, so we write

$$\begin{aligned}
\lim_{b\to\infty}\int_0^b t^n e^{-t}\ dt &= \lim_{b\to\infty}\left([-t^n e^{-t}]_0^b - \int_0^b -(n)t^{n-1}e^{-t}\ dt\right) \\
&= \lim_{b\to\infty}\left([-t^n e^{-t}]_0^b + \int_0^b (n)t^{n-1}e^{-t}\ dt\right) \\
&= \lim_{b\to\infty}\left([-t^n e^{-t}]_0^b + (n)\int_0^b t^{n-1}e^{-t}\ dt\right) \\
&= \lim_{b\to\infty}\left(-\frac{b^n}{e^b} - \left(-\frac{0^n}{e^0}\right) + n\Gamma(n)\right) \\
&= \lim_{b\to\infty}\left(-\frac{b^n}{e^b} + 0 + n\Gamma(n)\right)
\end{aligned}$$

The latter two terms of the limit are easy to take, but to take the first limit we once again need that $e^b$ dominates $b^n$ for any $n$. We'll see a rigorous reason why when we study sequences, but for now we'll just accept that $\lim\limits_{b\to\infty}\frac{b^n}{e^b} = 0$ because that will get us the right answer.

Anyway, we find $\Gamma(n+1) = n\Gamma(n)$ as we hoped.

(c) We need a function that satifies $f(1) = 1$ and $f(2) = 1$ and $f(n+1) = n \cdot f(n)$. We can calculate some values of $f$:

| $n$ | $f(n)$ |
|---|---|
| 1 | 1 |
| 2 | 1 |
| 3 | $2 \cdot f(2) = 2 \cdot 1 = 2$ |
| 4 | $3 \cdot f(3) = 3 \cdot 2 \cdot 1 = 6$ |
| 5 | $4 \cdot f(4) = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ |

This looks almost like the factorial function, $f(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$. The difference is $2! = 2 \cdot 1 = 2$, not 1, so we have

$$\Gamma(n+1) = n!$$