MATH 107-153 Recitation 8-9

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p. 351 #15, 16: Consider the two integrals

$$\int x^2 e^{3x}\ dx \quad \textbf{and} \quad \int x^2 e^{x^3}\ dx\ .$$

Both of these integrals can be solved using tools you already have, but one is much nicer than the other.

1. **Identify and solve the easier integral.**
2. **Come up with a "plan of attack" for the harder integral. In other words, decide which integration technique you would use to solve it and, if relevant, what subsitutions you would make (such as choosing a u for u-substitution or a u and a dv for integration by parts.**
3. **Instead of solving the integral that way, find an applicable rule in a table and use that to find the integral. Can you see how your technique would have lead to the correct answer?**

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1. The integral on the right allows for a nice u-substitution. Let $u = x^3$. Then $\frac{du}{dx} = 3x^2$ so $dx = \frac{du}{3x^2}$ and

$$\begin{aligned}\int x^2 e^{x^3}\ dx &= \int \cancel{x^2} e^u \frac{du}{3\cancel{x^2}} \\ &= \frac{1}{3}\int e^u\ du \\ &= \frac{1}{3}e^u + C \\ &= \frac{1}{3}e^{x^3} + C\end{aligned}$$

2. Since no obvious substitution is present, let's try integration by parts. Both x^2 and e^{3x} are easy to differentiate or integrate, so let's choose to differentiate x^2 to make it simpler. Then

$$\begin{aligned} u &= x^2 & dv &= e^{3x}dx \\ du &= 2xdx & v &= \tfrac{1}{3}e^{3x}\end{aligned}$$

so

$$\int x^2 e^{3x}\ dx = \frac{1}{3}x^2 e^{3x} - \int \frac{2}{3}xe^{3x}\ dx.$$

While we still don't have a basic integration rule to use, the new integral is simpler because the degree of x went down. It looks like we could try integration by parts again to get the answer.

3. Looking at rule III. 14 in the back of the book, we find

$$\int p(x)e^{ax}\ dx = \frac{1}{a}p(x)e^{ax} - \frac{1}{a^2}p'(x)e^{ax} + \frac{1}{a^3}p''(x)e^{ax} - \ldots + C$$

where $p(x)$ is a polynomial. Plugging in this formula, we get

$$\begin{aligned}\int x^2e^{3x}\ dx &= \frac{1}{3}x^2e^{3x} - \frac{1}{9}\cdot 2xe^{3x} + \frac{1}{27}\cdot 2e^{3x} - \frac{1}{81}\cdot 0e^{3x} + C \\ &= e^{3x}\left(\frac{x^2}{3} - \frac{2x}{9} + \frac{1}{27}\right) + C\end{aligned}$$

It's easy to see where this formula comes from. All it is doing is repeatedly applying integration by parts until we've taken the derivative of the polynomial so many times that it becomes zero. We also have to add an additional $\frac{1}{a}$ factor for every derivative because we're taking the antiderivative of e^{ax} an additional time.

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p. 351 #34: Solve the integral

$$\int \frac{1}{y^2+4y+5}\ dy$$

using the following steps:

1. **Look at a table of integrals and decide on the form**

$$\int \frac{1}{x^2+a^2}\ dx = \frac{1}{a}\arctan\frac{x}{a} + C, a \neq 0.$$

2. **Use the "completing the square" technique to get the integral into the appropriate form.**

3. **Apply the ingegration rule you chose in step 1.**

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1. It really does appear that this form is the best choice. Becuase we're adding everything in the denominator, it is unlikely we can factor $y^2 + 4y + 5 = (y - a)(y - b)$ so the only choice with y^2 in the denominator not under a square root is the form above.

2. Remember that in completing the square we want to get an expression of the form $(y + a)^2 + b$ that is equal to $y^2 + 4y + 5$. We know that $(y + a)^2 = y^2 + 2ay + a^2$ and we want $2ay = 4y$ so let's choose $a = 2$. But $(y + 2)^2 = y + 4y + 4$, so we have 1 left over; we see $b = 1$. In short, $y^2 + 4y + 5 = (y + 2)^2 + 1$.

3. Now we can rewrite the formula:

$$\int \frac{1}{y^2+4y+5}\ dy = \int \frac{1}{(y+2)^2+1}\ dy$$

and we can use the integration rule with $x = (y + 2)$ and $a = 1$ (note that $a^2 = 1^2 = 1$) to get

$$\int \frac{1}{(y+2)^2+1}\ dy = \arctan(y+2) + C$$

3: Let a and b be distinct real numbers. Use partial fraction decomposition to find a simpler form of $\frac{1}{(x-a)(x-b)}$. Then use this decomposition to find

$$\int \frac{1}{(x-a)(x-b)}\ dx.$$

Finally, compare your answer to formula V. 26 in the back of the book.

Let

$$\frac{1}{(x-a)(x-b)} = \frac{M}{x-a} + \frac{N}{x-b}$$

for some constants M and N that we will figure out in a moment. Then we can multiply both sides by the denominator to get

$$\frac{\cancel{(x-a)(x-b)}}{\cancel{(x-a)(x-b)}} = \frac{M\cancel{(x-a)}(x-b)}{\cancel{(x-a)}} + \frac{N(x-a)\cancel{(x-b)}}{\cancel{(x-b)}} \text{ which simplifies to } 1 = M(x-b) + N(x-a)$$

There are two ways of finding M and N. The first is to note that the polynomial on the left has to be equal to the polynomial on the right which can only happen if the coefficients of each power of x are equal. In other words, we can rewrite the equation as

$$0 * x^1 + 1 * x^0 = (M+N)x^1 + (-aM - bN) * x^0$$

so

$$M + N = 0 \text{ and } -bM - aN = 1$$

We can solve this sysyem of equations to find $M = \frac{1}{a-b}$ and $N = \frac{1}{b-a}$.

Alternatively, note that we assume

$$\frac{1}{(x-a)(x-b)} = \frac{M}{(x-a)} + \frac{N}{(x-b)}$$

holds except when $(x-a)(x-b) = 0$ as then the fraction is undefined. However, we want

$$1 = M(x-b) + N(x-a)$$

to hold for every value of x, even if that value of x would make the first equation undefined. The first equation is undefined when $x = a$ and $x = b$ so let's make sure the second equation still holds then. That means

$$1 = M(a-b) + N(a-a) \text{ or } 1 = M(a-b) \text{ so } M = \frac{1}{a-b}$$

and

$$1 = M(b-b) + N(b-a) \text{ or } 1 = N(b-a) \text{ so } N = \frac{1}{b-a}.$$

Whichever method we choose, we can now solve the integral:

$$\begin{aligned}
\int \frac{1}{(x-a)(x-b)}\ dx &= \int \frac{\frac{1}{(a-b)}}{(x-a)} + \frac{\frac{1}{(b-a)}}{(x-b)}\ dx \\
&= \int \frac{\frac{1}{(a-b)}}{(x-a)}\ dx + \int \frac{\frac{1}{(b-a)}}{(x-b)}\ dx \\
&= \frac{1}{(a-b)} \int \frac{1}{(x-a)}\ dx + \frac{1}{(b-a)} \int \frac{1}{(x-b)}\ dx
\end{aligned}$$

From here you may be able to see the answer, but technically we need to use u substitution to apply our basic antidifferentiation rules. So let $u = x - a$ and $du = dx$ and also let $w = x - b$ and $dw = dx$ to get

$$\begin{aligned}
\frac{1}{(a-b)} \int \frac{1}{(x-a)}\ dx + \frac{1}{(b-a)} \int \frac{1}{(x-b)}\ dx &= \frac{1}{(a-b)} \int \frac{1}{u}\ du + \frac{1}{(b-a)} \int \frac{1}{w}\ dw \\
&= \frac{1}{a-b} \ln|u| + \frac{1}{b-a} \ln|w| + C \\
&= \frac{1}{a-b} \ln|x-a| + \frac{1}{b-a} \ln|x-b| + C \\
&= \frac{1}{a-b} (\ln|x-a| - \ln|x-b|) + C
\end{aligned}$$

which matches formula V. 26 exactly.