MATH 107-153 Recitation 6-7

JD Nir
Avery 230 • Office Hours: W 4-5 R 11-2
jnir@huskers.unl.edu
www.math.unl.edu/∼jnir2/107-153.html
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p. 346 #14: Find $\int \cos^2(3\alpha + 1)\ d\alpha$.

This is a tricky one. If we remember the double angle formulas of trig functions, we could rewrite $\cos^2(3\alpha+1)$ in an easier form. But let's try to approach this with integration by parts.

First let's try the obvious case: Let $u = \cos^2(3\alpha + 1)$ and $dv = 1\ d\alpha$. Then $du = 2\cos(3\alpha+1) \cdot -\sin(3\alpha+1) \cdot 3\ d\alpha = 6\sin(3\alpha+1)\cos(3\alpha+1)\ d\alpha$ and $v = \alpha$. Then

$$\int \cos^2(3\alpha+1)\ d\alpha = (\cos^2(3\alpha+1))(\alpha) - \int (\alpha)(6\sin(3\alpha+1)\cos(3\alpha+1)\ d\alpha)$$

We can *almost* do the new integral with a substitution (by letting $w = \sin(3\alpha+1)$ so $d\alpha = \frac{dw}{3\cos(3\alpha+1)}$ but we will have an α left over that will be difficult to deal with.

Instead, let's try a more clever choice for u and dv. Let $u = \cos(3\alpha+1)$ and $dv = \cos(3\alpha+1)\ d\alpha$. Then $du = -3\sin(3\alpha+1)\ d\alpha$ and $v = \frac{\sin(3\alpha+1)}{3}$. This gives us

$$\begin{aligned}\int \cos^2(3\alpha+1)\ d\alpha &= \frac{1}{3}\sin(3\alpha+1)\cos(3\alpha+1) - \int \frac{\sin(3\alpha+1)}{3} \cdot -3\sin(3\alpha+1)\ d\alpha \\ &= \frac{1}{3}\sin(3\alpha+1)\cos(3\alpha+1) + \int \sin(3\alpha+1)^2\ d\alpha\end{aligned}$$

The new integral looks just as hard as the last one! In fact, it looks almost the same. Recall that $\sin^2\theta + \cos^2\theta = 1$ for any angle θ. So we can rewrite the integral as

$$\int \sin(3\alpha+1)^2\ d\alpha = \int 1 - \cos^2(3\alpha+1)\ d\alpha = \alpha - \int \cos^2(3\alpha+1)\ d\alpha.$$

Now we have the *exact* problem we started with. Let's treat the whole integral as a variable and combine like terms:

$$\begin{aligned}\int \cos^2(3\alpha+1)\ d\alpha &= \frac{1}{3}\sin(3\alpha+1)\cos(3\alpha+1) + \alpha - \int \cos^2(3\alpha+1)\ d\alpha \\ 2\int \cos^2(3\alpha+1)\ d\alpha &= \frac{1}{3}\sin(3\alpha+1)\cos(3\alpha+1) + \alpha.\end{aligned}$$

Then all we need to do is divide by two to get our answer. Don't forget the constant of integration!

$$\int \cos^2(3\alpha+1)\ d\alpha = \frac{1}{6}\sin(3\alpha+1)\cos(3\alpha+1) + \frac{\alpha}{2} + C.$$

p. 346 #14: Find $\int (\ln t)^2 \; dt$.

Let's try integration by parts using $u = (\ln t)^2$ and $dv = 1 \; dt$. Then $du = 2\ln t \cdot \frac{1}{t} \; dt$ and $v = t$. This gives us

$$\int (\ln t)^2 \; dt = ((\ln t)^2)(t) - \int (t)(2\ln t \cdot \frac{1}{t} \; dt) = t(\ln t)^2 - 2\int \ln t \; dt.$$

We're stuck with $\int \ln t \; dt$, which we don't have a rule for, but we can try using integration by parts again. This time let $u' = \ln t$ and $dv' = 1 \; dt$. Then $du' = \frac{1}{t} \; dt$ and $v' = t$. This gives us

$$\int \ln t \; dt = (\ln t)(t) - \int (t)(\frac{1}{t} \; dt) = t\ln t - \int dt.$$

Finally we get an easy integral! Plugging this all back in, we get

$$t(\ln t)^2 - 2\int \ln t \; dt = t(\ln t)^2 - 2\left(t\ln t - \int dt\right) = t(\ln t)^2 - 2t\ln t + 2t + C.$$

p. 346 #28: Find $\int x^5 \cos x^3 \; dx$.

This almost looks like a u-substitution problem. Let's try $u = x^3$ and see where that gets us. Then $dx = \frac{du}{3x^2}$ so

$$\int x^5 \cos x^3 \; dx = \int \frac{x^5}{3x^2} \cos u \; du = \frac{1}{3}\int x^3 \cos u \; du.$$

Well, not all of the powers of x canceled, but we are left with $x^3 = u$. So we can rewrite the integral as

$$\frac{1}{3}\int x^3 \cos u \; du = \frac{1}{3}\int u \cos u \; du.$$

While we don't have a rule for this kind of integral, it looks much easier to do with integration by parts! Lets use w instead of u so we don't get confused with our u-substitution. Choose $w = u$ and $dv = \cos u \; du$. Then $dw = du$ and $v = \sin u$. So

$$\frac{1}{3}\int u \cos u \; du = \frac{1}{3}\left(u \sin u - \int \sin u \; du\right) = \frac{1}{3}(u \sin u + \cos u) + C$$

Now, of course, we have to go back to the u-substitution and use x instead, giving us our final answer:

$$\int x^5 \cos x^3 \; dx = \frac{1}{3}(x^3 \sin x^3 + \cos x^3) + C.$$

p. 346 #49: Use integration by parts twice to find $\int e^x \sin x\ dx$.

Both e^x and $\sin x$ are pretty easy to both differentiate and anti-differentiate which makes it pretty hard to choose values for u and dv. Let's start with $u = e^x$ and $dv = \sin x\ dx$. Then $du = e^x\ dx$ and $v = -\cos x$ so

$$\int e^x \sin x\ dx = (e^x)(-\cos x) - \int (-\cos x)(e^x\ dx) = -e^x \cos x + \int e^x \cos x\ dx.$$

Well, this didn't work, but the problem said we'd need integration by parts twice, so let's just dot it again. Let $u' = e^x$ and let $dv' = \cos x\ dx$. Then $du' = e^x dx$ and $v' = \sin x$ so

$$-e^x \cos x + \int e^x \cos x\, dx = -e^x \cos x + (e^x)(\sin x) - \int (\sin x)(e^x\ dx) = e^x \sin x - e^x \cos x - \int e^x \sin x\, dx$$

It looks like we haven't gotten anywhere, but remember where we started!

$$\begin{aligned} \int e^x \sin x\ dx &= e^x \sin x - e^x \cos x - \int e^x \sin x\ dx \\ 2 \int e^x \sin x\ dx &= e^x \sin x - e^x \cos x \\ \int e^x \sin x\ dx &= \frac{e^x \sin x - e^x \cos x}{2} + C \end{aligned}$$

It looks like we chose the correct u and dv in the first step. I wonder what would have happened if we'd switched them?

Let $u = \sin x$ and $dv = e^x\ dx$. Then $du = \cos x\ dx$ and $v = e^x$ so

$$\int e^x \sin x\ dx = (\sin x)(e^x) - \int e^x \cos x\ dx.$$

Using integration by parts again with $u' = \cos x$ and $dv' = e^x\ dx$ we get $du' = -\sin x\ dx$ and $v' = e^x$ so

$$(\sin x)(e^x) - \int e^x \cos x\, dx = e^x \sin x - \left((\cos x)(e^x) - \int (e^x)(-\sin x\ dx) \right) = e^x \sin x - e^x \cos x - \int e^x \sin x\, dx.$$

We can do the same trick to combine the $\int e^x \sin x\ dx$ and we get the same answer.

In this case it doesn't matter what we choose for u and dv as long as we stay consistent both times we use integration by parts. What happens when you use $u = e^x$ and $dv = \sin x\ dx$ but then use $u' = \cos x$ and $dv' = e^x\ dx$?