MATH 107-153 Recitation 16

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1: Find the volume of the solid whose base is the region in the first quadrant bounded by $y = x^2$, $y = 1$, and the y-axis and whose cross-sections perpendicular to the x-axis are semicircles.

Break up the solid into semi-circular prisms of width Δx. Each semi-circle has radius $\frac{1}{2}(1 - x^2)$ so the semi-circular prism has volume $\frac{1}{2}\pi(1 - \frac{1}{2}x^2)^2\Delta x = \frac{\pi}{8}(1 - 2x^2 + x^4)\Delta x$. Adding up these estimations gives

$$\sum_{i=1}^{n} \frac{\pi}{8}(1 - 2x_i^2 + x_i^4)\Delta x$$

and taking the limit gives

$$\int_0^1 \frac{\pi}{8}(1 - 2x^2 + x^4)\ dx = \frac{\pi}{8}\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1 = \frac{\pi}{15}.$$

2: Find the equation (in terms of x and y) of the tangent line to the curve $r = 2\sin 4\theta$ at $\theta = \frac{\pi}{3}$.

In order to find the equation of a line, we need two things: a point (in Cartesian coordinates) and a slope ($\frac{dy}{dx}$). Then we can use the point-slope definition of a line.

We know the point we are interested in, albeit in polar coordinates. When $\theta = \frac{\pi}{3}$, we have $r = 2\sin(\frac{4\pi}{3}) = -\sqrt{3}$ so our point in polar coordinates is $(\sqrt{3}, \frac{\pi}{3})$. Translating this into cartesian coordinates gives us

$$(x, y) = (r\cos\theta, r\sin\theta) = (\sqrt{3}\cos\left(\frac{\pi}{3}\right), \sqrt{3}\sin\left(\frac{\pi}{3}\right)) = (-\tfrac{\sqrt{3}}{2}, -\tfrac{3}{2}).$$

To find $\frac{dy}{dx}$, we note that

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

and as we have $x = r\cos\theta = 2\sin(4\theta)\cos\theta$ and $y = r\sin\theta = 2\sin(4\theta)\sin\theta$ we can calculate

$$\frac{dy}{d\theta} = 2\sin(4\theta)\cos\theta + 8\cos(4\theta)\sin\theta \text{ and } \frac{dx}{d\theta} = -2\sin(4\theta)\sin\theta + 8\cos(4\theta)\cos\theta.$$

Evaluating that ratio at $\theta = \frac{\pi}{3}$ gives

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = 5\sqrt{3}.$$

Now using point-slope form we find

$$y - (-\tfrac{3}{2}) = 5\sqrt{3}(x - (-\tfrac{\sqrt{3}}{2}))$$

which gives us the equation $y = 5\sqrt{3}(x + \frac{\sqrt{3}}{2}) - \frac{3}{2}$.

3: A storage shed is the shape of a half-cylinder of radius r and length l.

(a) What is the volume of the shed?

(b) The shed is filled with sawdust whose density (mass/unit volume) at any point is proportional to the distance of that point from the floor. The constant of proportionality is k. Calculate the total mass of sawdust in the shed.

(a) We should start by drawing a picture, but three-dimensional pictures are difficult to render on a computer, so reference the question in the WebWork to see an appropriate image. To find the volume, we will take horizontal slices of height Δy. This gives rectangular prisms with height Δy and length L, but we will have to find the width as a function of y. Using similar triangles, we find the width is $2\sqrt{r^2 - y^2}$ giving us a volume of $2l\sqrt{r^2 - y^2}\Delta y$ for our slice. This gives us the Riemann sum

$$\sum_{i=1}^{n} 2l\sqrt{r^2 - y_i^2}\Delta y.$$

Taking the limit, and noting that y ranges from 0 to r, we get the total volume is:

$$\int_0^r 2l\sqrt{r^2 - y^2}\ dy.$$

This is a difficult integral to take, so instead we fall back on geometry. The base of the prism is a semi-circle which has area $\frac{1}{2}\pi r^2$. Thus the volume of the prism is $\frac{1}{2}\pi r^2 l$.

(b) We set up the problem similar to last time, but now instead of finding the volume of our prism we want the mass of the sawdust in that prism. We know mass is volume times density. The density is proportional to the height, so $\delta(y) = ky$, so the mass of our prism is $2lky\sqrt{r^2 - y^2}\Delta y$ giving us the Riemann sum

$$\sum_{i=1}^{n} 2lky_i\sqrt{r^2 - y_i^2}\Delta y.$$

Taking the limit, and noting that y ranges from 0 to r, we get the total volume is:

$$\int_0^r 2lky\sqrt{r^2 - y^2}\ dy.$$

This integral we can solve be using the u-substitution $u = r^2 - y^2$. Doing so gives

$$\int_0^r 2lky\sqrt{r^2 - y^2}\ dy = lk\int_{u=0}^{u=r^2} u^{\frac{1}{2}}\ du = \frac{2}{3}Lkr^3.$$

4: A bucket of water of mass 20 kg is pulled at constant velocity up to a platform 50 meters above the ground. This takes 20 minutes, during which time 4 kg of water drips out at a steady rate through a hole in the bottom. Find the work needed to raise the bucket to the platform. (Use $g = 9.8\frac{m}{s^2}$.)

To analyze this problem, consider the work it will take to raise the bucket Δh when it is at height h from the ground. To do this, we will need to know the force it takes to overcome gravity, and to know that we will need to know the weight of the bucket at height h. We will assume the bucket has constant weight while it is being raised Δh.

Note that 20 minutes is 1200 seconds, so the rate that water is dripping out is $\frac{4}{1200} = \frac{1}{300}\frac{\text{kg}}{\text{s}}$. Since the velocity used to pull up the bucket is constant, it must be $\frac{50}{1200} = \frac{1}{24}\frac{\text{m}}{\text{s}}$. Thus at height h, we have been pulling for $24h$ seconds and have lost $\frac{24h}{300} = \frac{2}{25}h$ kg of water, meaning the weight is $20 - \frac{2}{25}h$ kg. Thus the force needed to overcome gravity is $9.8(20 - \frac{2}{25}h)\Delta h$.

Adding up all of our approximations gives us the Riemann sum

$$\sum_{i=1}^{n} 9.8(20 - \frac{2}{25}h_i)\Delta h.$$

Taking the limit as $\Delta h \to 0$, and noting that h starts at 0 and goes to 50, we get

$$\text{Work} = \int_0^{50} 9.8(20 - \frac{2}{25}h)\ dh = 9.8\left[20h - \frac{1}{25}h^2\right]_0^{50} = 8820J.$$