My research is in commutative algebra, and I have a particular interest in criteria for detecting complete intersection rings. Behind polynomial rings, complete intersections are perhaps the second best class of rings; they are rings which can be expressed as “nice” quotients of polynomial rings. Polynomials, a fundamental object of study in algebra, have a counterpart in geometry in the visual properties of their graphs. Over the complex numbers, a connection between algebra and geometry is via the Fundamental Theorem of Algebra: A monic polynomial is determined precisely by its roots. Complete intersection rings also have a geometric analog, which is in terms of algebraic varieties and hypersurfaces. An algebraic variety is said to be a complete intersection if it is the intersection of the “right” number of hypersurfaces.

My interest is primarily in criteria for detecting the complete intersection property of local rings, that is, rings having a unique maximal ideal. In this context, Assmus [Ass59] provided criteria in terms of a complex associated with the ring, as opposed to focusing only on the desired structure of the ring as a “nice” quotient of a polynomial ring. Computationally, the criteria took two distinct forms: the vanishing of a single homology object and the vanishing of an infinite family of homology objects. Using the computer algebra system Macaulay2 [GS], and in particular the package DGAlgebras [Moo], the former criterion is easily verified. Predictably, the latter cannot be (directly) computationally verified. Relaxing the requirement that the ring be local, this second criterion took on new life in the work of Blanco, Majadas, and Rodicio, who showed in [BMR98] that it is equivalent to the defining property of quasi-complete intersection ideals. The terminology is sensible, as this property generalizes the complete intersection property of rings: Assmus’ result in [Ass59] yields that a local ring is a complete intersection precisely when its maximal ideal is a quasi-complete intersection.

My research provides an effective analog for Assmus’ first criterion in the case of quasi-complete intersections, thus allowing for a finite computational process for detecting quasi-complete intersection ideals. In addition, I study the complex appearing in Assmus’ characterization of complete intersections, and I identify in more detail its structure when the ring is not a complete intersection. This project also yields computational benefits, as it reveals that when a ring is not a complete intersection, there are infinitely many non-vanishing homology objects.

In Section 1, I’ll describe in more detail the relevant background and establish the context for my research. I’ll present the main results of my thesis in Section 2. My continuing efforts and future plans to improve and expand on the results of my thesis are outlined in Section 3. I am excited to include bright young students in my research; I will highlight my specific plans for future projects with undergraduates in Section 4.

1 Background

Given an ideal \( I \) over a commutative ring \( R \) and a generating set \( f = \{f_1, \ldots, f_n\} \) of \( I \), the Koszul complex on a \( f \), denoted \( K(f) \), represents a first step in building a resolution of the quotient \( R/I \). Koszul complexes are ubiquitous in the homological study of commutative rings, and the homological structure of \( K(f) \) encodes deep insights about the algebraic structure of \( I \). For example, when \( R \) is Noetherian, the highest degree in which the homology \( H_i(K(f)) \) is non-zero

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1The nomenclature is due to Avramov, Henriques, and Sega [AHS13]. As Avramov, et al. note, quasi-complete intersection ideals were first introduced in Rodicio’s paper [Rod95] and in his joint work with Blanco and Majadas [BMR96] as ideals having free exterior Koszul homology.
reveals the \textit{depth} of \( I \), an invariant of \( I \) which gives a measure of how many elements of \( I \) act like linearly independent variables.

When \( R \) is a (commutative, Noetherian) local ring, the homology of each Koszul complex splits cleanly into a low-degree portion of non-zero homology and a high-degree portion with zero homology. The homology of a Koszul complex on a generating set \( x \) of the maximal ideal detects whether \( R \) has the structure of a polynomial ring; this occurs precisely when \( K(x) \) is acyclic, that is, when \( H_i(K(x)) = 0 \) for all \( i > 0 \). Even better, to check in general (over a local ring) whether \( K(f) \) is acyclic, it is enough to check whether \( H_1(K(f)) \) is zero. When this occurs, we call the ideal \( I = (f) \) a \textit{complete intersection ideal}, and it can be minimally generated by a sequence of linearly independent elements.

If a Koszul complex represents a first step in building a resolution, then the subsequent steps, introduced by Tate [Tat57], become necessary when \( H_1(K(f)) \) is non-zero. Tate’s construction yields a new complex, called the \textit{two-step Tate complex}, which we denote by \( T \). It is obtained by adjoining new variables to the Koszul complex to annihilate the problematic degree one homology, making \( T \) (at least) one step closer to being a resolution of the quotient. This construction depends on the choice of a set of cycles \( z \) whose homology classes generate \( H_1(K(f)) \). When \( T \) is acyclic (and thus is a resolution of \( R/I \)), no further adjunctions are necessary, and a Theorem of Blanco, Majadas, and Rodicio [BMR98] yields that that \( I \) is a quasi-complete intersection ideal. Assmus [Ass59] proves that a local ring is a complete intersection precisely when the maximal ideal is a quasi-complete intersection, so that \( T \) is acyclic. Even better, Assmus’ result also establishes that to check whether \( T \) is acyclic, it is enough to check whether \( H_2(T) = 0 \).

\section{Main Results}

The main results of my dissertation take the form of a give-and-take in the detection of the quasi-complete intersection property: by assuming more about the structure of \( I \), we can assume less about the vanishing of the homology of \( T \) to conclude that \( I \) is a quasi-complete intersection ideal. On one end of the spectrum, I show in Theorem 1 that for finitely generated ideals, the quasi-complete intersection property can be detected from a low-degree band of vanishing of \( H_*(T) \). On the other end of the spectrum, we assume much more about the ideal, namely that it is the unique maximal ideal of a local ring, and, in Theorem 3, I improve on results of Assmus [Ass59] and Tate [Tat57].

The setup is as follows: \( R \) is a commutative ring, \( f = \{f_1, \ldots, f_n\} \) is a sequence of elements in \( R \) which generate the proper ideal \( I \), and \( T \) is the two-step Tate complex associated with \( K(f) \) and a finite set of cycles \( z \) whose homology classes generate \( H_1(K(f)) \).

\textbf{Theorem 1} ([Lut15]). Set \( b = \max\{i : H_i(K(f)) \neq 0\} \). Then \( I \) is a quasi-complete intersection if and only if \( H_i(T) = 0 \) for \( i = 1, 2, \ldots, b + 2 \).

With an additional assumption on the structure of \( H_1(K(f)) \), namely the size of a generating set, I show that there is flexibility in the location of the band of vanishing of \( H_*(T) \):

\textbf{Theorem 2} ([Lut15]). Suppose the set of cycles \( z \) consists of \( b \) elements, \( z = \{z_1, \ldots, z_b\} \), where \( b \) as in Theorem 1. Then \( I \) is a quasi-complete intersection if and only if there exists an integer \( k \geq 1 \) with

\[ H_k(T) = H_{k+1}(T) = \cdots = H_{k+b}(T) = 0. \]
This additional hypothesis allows for an improvement on Theorem 1 in two respects: we have flexibility on the location of the band of vanishing of $H_\ast(T)$, and the width of the band is decreased to $b + 1$ (from $b + 2$). In addition, this characterizes finitely generated quasi-complete intersection ideals as ideals for which $H_\ast(T)$ eventually vanishes.

We now consider the other end of the spectrum: the local case. In what follows, $R$ is a local (Noetherian) ring with unique maximal ideal $m$. Let $x$ denote a minimal generating set of $m$ and let $z = \{z_1, \ldots, z_m\}$ denote a set of cycles whose homology classes form a minimal generating set of $H_1(K(x))$. Let $T$ be the two-step Tate complex associated with $K(x)$ and $z$. The classical characterization of complete intersection rings in this context is the following:

**Theorem ([Ass59],[Tat57]).** The following conditions are equivalent:

1. $R$ is a complete intersection ring,
2. $H_\ast(T) = 0$ for all $i > 0$,
3. $H_2(T) = 0$.

I complement the previous result as follows:

**Theorem 3 ([Lut15]).** The following conditions are equivalent:

1. $R$ is a complete intersection ring,
2. There exists an integer $k \geq 0$ such that $H_k(T) = H_{k+1}(T) = \cdots = H_{k+m-1}(T) = 0$,
3. $H_i(T) = 0$ for some $i \in \{3, 4\}$.

Previous work and the work of my thesis suggests that condition (3) of Theorem 3 can be improved to $i \geq 2$, so that the complete intersection property can be detected from the vanishing of $H_\ast(T)$ in a single degree ($\geq 2$). In particular, I show in the following result that this improvement holds for large class of local rings:

**Theorem 4 ([Lut15]).** Suppose there is a Golod map from a complete intersection ring onto $R$. Then $R$ is a complete intersection if and only if $H_i(T) = 0$ for some $i \geq 2$.

Among the rings which satisfy the hypothesis of Theorem 4 are Golod rings, which are defined by an extremal property on the ranks of free modules appearing in a minimal free resolution of the residue field $R/m$. By a result of Avramov, Kustin, and Miller [AKM88], a ring which is a quotient of a polynomial ring in 4 or fewer variable satisfies the hypothesis. A result of Jacobsson, Kustin, and Miller [JKM85] yields that the hypothesis is also satisfied by any Gorenstein ring which is quotient of a polynomial ring in 5 variables.

## 3 Current and Future Work

I am interested in extending the results of Theorem 4. Phrased as a question about the structure of $H_\ast(T)$ for rings which are not complete intersections, we have the following:

**Question 5.** Is it the case that any ring which is not a complete intersection has $H_i(T) \neq 0$ for all $i \geq 2$?

In a similar vein, I would like to know whether the result of Theorem 2 is optimal:
Question 6. In the situation of Theorem 2, is there an ideal which is not a quasi-complete intersection and satisfies

\[ H_k(T) = H_{k+1}(T) = \cdots = H_{k+b-1}(T) = 0 \]

for some \( k \geq 1 \)?

An affirmative answer to this question would imply that the result of Theorem 2 is optimal.

4 Collaborative Research with Undergraduates

My commitment to actively engaging students in my mathematics classrooms extends to an enthusiasm for mentoring research projects with undergraduates, and my research fits well with traditional undergraduate mathematics courses. My ongoing projects present opportunities to include undergraduate students in research in commutative algebra. In addition, motivated students would be introduced to the rich theory of differential graded (DG) algebras and homological algebra.

Portions of my research utilize Macaulay2 [GS], a modern, actively-developed computer algebra system. The following are excellent projects for undergraduates, and each provides a concrete entry-point for undergraduates interested in commutative algebra. These projects also emphasize the programming aspect of work in computational algebra.

The following project would contribute to the investigation of Question 5:

Project 1. Design examples of quotients of polynomial rings in 5 variables which are not Gorenstein (and thus not complete intersections), and use the Macaulay2 package DGAlgebras [Moo] to identify potential patterns in the structure of \( H^*(T) \).

Currently, the DGAlgebras package only includes algorithms for building the Tate complex \( T \) in the local case. The next project seeks to extend the package’s functionality.

Project 2. Design an algorithm in Macaulay2 to build a two-step Tate construction associated with a Koszul complex \( K(f) \) for an arbitrary finite sequence \( f \).

Given a positive outcome of Project 2, Macaulay2 can be used to investigate Question 6, starting with a small number of variables:

Project 3. Design non-maximal ideals of (quotients of) polynomial rings in 2 variables which are not quasi-complete intersections and use the Macaulay2 package DGAlgebras [Moo] to identify potential patterns in the structure of \( H^*(T) \).

The proofs of Theorems 2 and 3 utilize periodicity properties of \( H_n(T) \). These properties are part of a larger set of results in [Lut15] on more general differential graded algebras, and their implementation in the DGAlgebras package would improve the efficiency of the existing algorithms.

Project 4. Incorporate the periodicity results of [Lut15] into the existing algorithms of the DGAlgebras [Moo].
Independent of these projects, I would like to design a “Topics” style course on the subject of computational algebra with a focus on the theory and applications of Gröbner bases. Whereas linear algebra provides tools for solving systems of linear equations, Gröbner bases provide algorithmic tools which solve systems of polynomial equations. The mathematical games of Sudoku and KenKen are wonderful examples in this area: these are games for which the solutions can be computed using Gröbner bases, and they serve as excellent motivating examples in computational algebra and the use of Macaulay2.

References


