Example 4.3. Let $\tilde{F} : \mathbb{C} \longrightarrow \mathbb{C}$ be defined by the following equation:

$$z' = (1 + 2\mathrm{i})z + (2 - \mathrm{i})\bar{z}$$

We denote $z = x + \mathrm{i}y$ and $z' = x' + \mathrm{i}y'$, where x, y, x', y' are real numbers (i.e. we write z and z' in algebraic form). Then we have:

$$\begin{aligned} z' = x' + \mathrm{i}y' &= (1 + 2\mathrm{i})(x + \mathrm{i}y) + (2 - \mathrm{i})(x - \mathrm{i}y) \\ &= x + \mathrm{i}y + 2\mathrm{i}x - 2y + 2x - 2\mathrm{i}y - \mathrm{i}x - y \\ &= 3x - 3y + \mathrm{i}(x - y) \end{aligned}$$

The plane transformation F associated with $\tilde{F}$ is defined by the following system of equations:

$$\begin{cases} x' = 3(x + y) \\ y' = x - y \end{cases} \tag{4.6}$$

Remark 4.4. (1) We can draw diagrams like diagram (4.1) for mappings from $\mathcal{P}$ to itself which are not invertible.

(2) Similar complex mappings are associated with mappings which are not defined on the whole of the plane, for example, inversions [1].

We intend to develop this in a subsequent note.

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Nonlocal advection equations

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Nonlocal problems are largely ignored in graduate and undergraduate texts on partial differential equations. Yet, nonlocal advection equations are important in many applications, and their solution provides an excellent illustration of the method of characteristics and how careful one must be to apply it.

1. Introduction

In this note we show how some simple, nonlocal advection equations can be resolved by the method of characteristics, and we present an example with an analytic solution and one with no solution. The analysis will show how these problems differ from their local counterparts, which, unlike nonlocal problems, have been thoroughly treated in various textbooks at all levels. The equations we consider are a subclass of partial-differential integral equations.

A simple thought experiment reveals how these nonlocal problems can occur. Let $u = u(x, t)$ be the density, or concentration, of some quantity (organisms, chemical, etc.) in a one-dimensional domain $0 < x < 1$, and let $q = q(x, t)$ be the flux of the quantity. Then, under smoothness assumptions, the basic conservation law is

$$u_t + q_x = 0$$

The simplest constitutive assumption is that the flux is proportional to the concentration, i.e. $q = cu$, where c is constant having dimensions of speed. Then we obtain the advection equation

$$u_t + cu_x = 0$$

which has simple travelling wave solutions $u = f(x - ct)$ of constant speed c. If the flux is $q = q(u)$, then we obtain the nonlinear, kinematic wave equation (the inviscid Burgers' equation):

$$u_t + c(u)u_x = 0, \quad c(u) = q'(u)$$

where the signal speed $c(u)$, not known *a priori*, depends upon the solution itself. This equation is the model for the development of weak solutions and shocks [1]. Both equations above are easily handled by the method of characteristics, which is developed in most elementary courses. In all of these models we assume negligible mixing in the x-direction and ignore diffusion.

In a different direction, it is not difficult to imagine that the speed that signals propagate depends in some manner upon the total mass in the system, i.e. the flux is given by a constitutive relation of the form, say,

$$q = v(t)u = \left(\int_0^1 F(u(x, t)) \, \mathrm{d}x \right) u \tag{1}$$

where F is a given function. For example, in the study of digestion processes in various organisms, it is believed that food moves through their gut, modelled as a plug flow reactor, at a speed depending upon net quality of food in the gut; low quality food moves through quickly, and high quality food moves slowly in order to get maximum absorption of nutrients. If u is a nutrient concentration, then one model of the flux that depends on an integrated nutrient concentration is given by equation (1). Then we obtain a nonlocal (because the unknown is integrated), nonlinear, advection equation

$$u_t + \left(\int_0^1 F(u(x, t)) \, \mathrm{d}x \right) u_x = 0$$

Equations similar to this, with appended initial, boundary, or signaling conditions, are the subject of this note.

There are many examples of nonlocal problems. Often the nonlocality is in the diffusion term (see Murray [2] for examples in biology). But, nonlocal advection equations occur as well. See Logan and Wolesensky [3] and Ledder and Logan [4] for examples in filtration theory, and see Cohn *et al.* [4] for an example in ion transport in electrolytic solutions, ignoring diffusion.

2. Initial value problem

A simple example shows the differences and difficulties encountered in analysing nonlocal models. Consider the class of model problems

$$u_t + \left(\int_0^1 F(u(x,t))\,\mathrm{d}x \right) u_x = 0, \quad 0 < x < 1,\ t > 0 \tag{2}$$

$$u(x,0) = f(x), \quad 0 < x < 1 \tag{3}$$

$$u(0,t) = 0, \quad t > 0 \tag{4}$$

where the concentration at the boundary is zero. We take the initial concentration f and the function F to be given nonnegative, smooth functions, and we assume $f(0) = 0$. The PDE (2) can be written

$$u_t + v(t)u_x = 0$$

where

$$v(t) = \int_0^1 F(u(x,t))\,\mathrm{d}x \tag{5}$$

the signal speed, is not known, but depends upon total mass in the system at time t. This problem can be resolved using the method of characteristics [6]. Defining characteristic curves as solutions of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v(t)$$

we have, on those curves,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u_t + u_x \frac{\mathrm{d}x}{\mathrm{d}t} = u_t + u_x v(t) = 0$$

Therefore u is constant on the characteristic curves. By simple integration,

$$x = \varphi(t) \equiv \int_0^t v(s)\,\mathrm{d}s$$

is the equation of the leading characteristic (signal) emanating from the origin (figure 1). Note that $v(t)$ and this characteristic are not yet known. In the region $\varphi(t) < x < 1$ (i.e. ahead of the leading signal), the characteristic emanating from $(\xi, 0)$ to (x,t) can be written $x = \varphi(t) + \xi$, and in the region $0 < x < \varphi(t)$ (i.e. behind the leading signal) the characteristic emanating from $(0,\tau)$ to (x,t) can be written

$$x = \int_\tau^t v(s)\,\mathrm{d}s = \varphi(t) - \varphi(\tau) \tag{6}$$

Behind the wave, in $0 < x < \varphi(t)$, the solution is $u(x,t) = u(0,\tau) = 0$, where $\tau = \tau(x,t)$ is the solution to $x = \varphi(t) - \varphi(\tau)$. Because u is constant on the characteristic connecting $(\xi, 0)$ to (x,t), we must have $u(x,t) = u(\xi,0) = f(\xi)$, $\varphi(t) < x < 1$. Thus, the solution ahead of the leading signal is

$$u(x,t) = f(x - \varphi(t)), \quad \varphi(t) < x < 1 \tag{7}$$

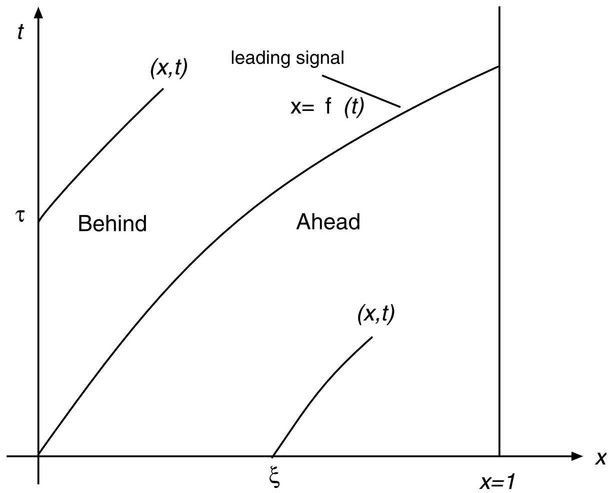

Figure 1. A generic space–time diagram showing the characteristics and the regions ahead and behind the leading wave front. The characteristic connecting $(\xi, 0)$ to (x, t) is given by $x = \varphi(t) + \xi$ and the characteristic connecting $(0, \tau)$ to (x, t) is given by equation (6).

but $\varphi(t)$ is still not determined. To find an equation for $v(t)$, and therefore φ, we write equation (5) as

$$v(t) = \int_{\varphi(t)}^{1} F(u(x, t))\,\mathrm{d}x$$

and we differentiate, using Leibniz rule, to get

$$\begin{aligned} v'(t) &= \int_{\varphi(t)}^{1} F'(u)u_t(x, t)\,\mathrm{d}x - u(\varphi(t), t)\varphi'(t) \\ &= -v(t)\int_{\varphi(t)}^{1} F'(u)u_x\,\mathrm{d}x - u(\varphi(t), t)\varphi'(t) \\ &= -v(t)\int_{\varphi(t)}^{1} \frac{\partial}{\partial x}F(u)\,\mathrm{d}x - u(\varphi(t), t)\varphi'(t) \\ &= -v(t)F(u(1, t)) \\ &= -v(t)(F \circ f)(1 - \varphi(t)) \end{aligned}$$

This is a differential–integral equation for $v(t)$; the initial condition is $v(0) = v_0 = \int_0^1 (F \circ f)(r)\,\mathrm{d}r$. But, because $v = \varphi'$, we can write

$$\varphi'' = -\varphi'(F \circ f)(1 - \varphi) \tag{8}$$

which is a second-order differential equation for φ.

This equation does not contain the independent variable t explicitly, and we can proceed by letting $w = \varphi'$. Whence

$$\varphi'' = w' = \frac{\mathrm{d}w}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = w\frac{\mathrm{d}w}{\mathrm{d}\varphi}$$

Then equation (8) becomes

$$\frac{\mathrm{d}w}{\mathrm{d}\varphi} = -(F \circ f)(1-\varphi)$$

and integration yields

$$w = w(0) - \int_0^{\varphi} (F \circ f)(1-r)\,\mathrm{d}r \tag{9}$$

where $w(0) = \varphi'(0) = v(0) = v_0$. This equation defines w as a function of φ. Now, φ can be determined as a function of t by quadrature from $\varphi' = w$, with the initial condition $\varphi(0) = 0$.

Equation (9) reveals several cases where there is chance of finding an analytic solution to the problem; for example, when F and f are inverse functions, the determination of w is simple.

Example. If $f(x) = x$ and $F(u) = u$, then $v_0 = \frac{1}{2}$, and

$$w = \frac{1}{2} - \int_0^{\varphi} (1-r)\,\mathrm{d}r = \frac{1}{2} + \frac{1}{2}\varphi^2 - \varphi$$

Consequently,

$$\int \frac{\mathrm{d}\varphi}{\frac{1}{2} + \frac{1}{2}\varphi^2 - \varphi} = t + C$$

The integration on the left-hand side is easily performed and we can solve for φ to obtain

$$\varphi(t) = \frac{t}{t+2}$$

Therefore the characteristic speed is

$$v(t) = \frac{2}{(2+t)^2}$$

The solution ahead of the leading signal is

$$u(x,t) = x - \frac{t}{t+2}, \quad \frac{t}{t+2} < x < 1$$

Here, the leading signal itself is asymptotic to $x = 1$.

3. Nonzero boundary data

Now we analyse a more difficult problem where nonzero signalling data g is given along $x = 0$. Thus, consider

$$u_t + v(t)u_x = 0, \quad 0 < x < 1,\ t > 0 \tag{10}$$

$$u(x,0) = f(x), \quad 0 < x < 1 \tag{11}$$

$$u(0,t) = g(t), \quad t > 0 \tag{12}$$

$$v(t) = \int_0^1 F(u(x,t))\,\mathrm{d}x \tag{13}$$

Interestingly, if $f(x) = 0$ and $F(0) = 0$, this problem does not have a solution. For example, take $F(u) = u$. Ahead of the leading signal we get $u = 0$. If we derive a differential equation for $v(t)$ as in the last section, then

$$v'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\varphi(t)} u(x,t)\,\mathrm{d}x = -v(t)\int_0^{\varphi(t)} u_x\,\mathrm{d}x + u(\varphi(t),t)\varphi'(t) = -v(t)g(t)$$

However, in this special case $v(0) = 0$, which forces $v(t) = 0$ for all t, giving a contradiction. So we cannot find a smooth solution that satisfies the boundary condition. Physically, we may look at it in this way—there is no mass in the system at $t = 0$, and no mass can ever get into the system at the left boundary because we are specifying the concentration at the boundary, not the flux.

Therefore, in equations (10)–(13) assume g and f are nonnegative and smooth, and $F(0) \neq 0$. Following the same ideas presented in the last section, we can calculate the solution ahead and behind the leading signal $x = \varphi(t)$, where $\varphi(t)$ is the yet unknown antiderivative of the speed $v(t)$. We get

$$u(x,t) = f(x - \varphi(t)), \quad \varphi(t) < x < 1 \tag{14}$$

and

$$u(x,t) = u(0,\tau) = g(\tau), \quad 0 < x < \varphi(t)$$

where $\tau = \tau(x,t)$ is defined implicitly through

$$x = \int_\tau^t v(s)\,\mathrm{d}s = \varphi(t) - \varphi(\tau)$$

Note that φ is increasing and hence invertible, and therefore

$$\tau = \varphi^{-1}(\varphi(t) - x)$$

Consequently,

$$u(x,t) = g(\varphi^{-1}(\varphi(t) - x)), \quad 0 < x < \varphi(t) \tag{15}$$

All that remains is to determine $v(t)$, and and hence $\varphi(t)$. Let us split the integral and (13) as

$$v(t) = \int_0^{\varphi(t)} F(u(x,t))\,\mathrm{d}x + \int_{\varphi(t)}^1 F(u(x,t))\,\mathrm{d}x, \quad t \leq \varphi^{-1}(1) \equiv t^*$$

Here, t^* is the time that the leading characteristic crosses $x = 1$. We may have $t^* = \infty$ if the leading signal is asymptotic to $x = 1$. Then, by Leibniz rule, we obtain (omitting details, which are similar to those leading to equation (8))

$$v'(t) = v(t)(F \circ g)(t) - v(t)(F \circ f)(1 - \varphi(t))$$

Therefore we have a second-order, nonlinear initial value problem for the leading signal,

$$\varphi''(t) = \varphi'(t)[(F \circ g)(t) - (F \circ f)(1 - \varphi(t))], \quad 0 < t \leq t^*$$
$$\varphi(0) = 0, \quad \varphi'(t) = v(0) \neq 0$$

At this point, when the problem has been reduced to a solution to an ordinary differential equation, we consider the problem solved by equations (14) and (15) on the domain $0 \leq x \leq 1$, $0 \leq t \leq t^*$. It is unlikely, of course, that φ can be found analytically and inverted to obtain a close-form solution, or a formula.

In the case that t^* is finite, we can solve the problem for $t > t^*$ by repeating the same procedure as above with the 'initial' data given by $u = u(x, t^*)$ along the time line $t = t^*$. And we can continue this procedure.

The reader might legitmately ask what happens if a source term, or a reaction term, appears on the right side of equation (2) or (10). Generally, because u is not constant on characteristics, these problems are harder to resolve analytically.

Rather than specify signalling data (equation (13)) at the boundary, we may want to specify the flux along the boundary. In this case equation (13) is replaced by (1)

$$v(t)u(0,t) = g(t), \quad t > 0 \tag{16}$$

Notice that $v(t)$ is an integral involving the density, so this condition reminds one of birth rate conditions in age-structured population models [7]. To illustrate how these problems may be handled, for simplicity assume zero initial data ($f(x) = 0$) and consider the problem defined by equations (10), (11), (16), and (13). Now we have

$$v'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\phi(t)} F(u(x,t))\,\mathrm{d}x = v(t)F(u(0,t)) = v(t)F\left(\frac{g(t)}{v(t)}\right)$$

which is a differential equation for $v(t)$.

We leave the reader an exercise, namely to resolve the case when the boundary condition takes the form $u_x(0,t) = g(t)$.

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Radius of convergence of a power series

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We derive two simple and memorizable formulas for the radius of convergence of a power series which seem to be appropriate for teaching in an introductory calculus course.