Approximate wave fronts in a class of reaction-diffusion equations

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Communications on Applied Nonlinear Analysis ***8****(2), 23-30 (2001)*

Abstract. Under special conditions an approximate wave front solution to a system of nonlinear reaction-diffusion type equations is obtained. An assumption of fast kinetics permits the system to be reformulated as a single nonlinear differential-integral equation of convolution type. If the convolution kernel has the form of a mollifier then the convolution is localized to obtain an infinite system of differential equations. In some cases the system can be truncated and solved to obtain approximate wave fronts. Examples of this method are presented, one with nonlinear convection and one with nonlinear diffusion.

1. Introduction

We consider the nonlinear system

$$u_t = N(u) + uv, \tag{1.1}$$

$$\varepsilon v_t = P(D)v - u + a, \quad 0 < \varepsilon \ll 1. \tag{1.2}$$

where $u = u(x,t)$ and $v = v(x,t)$, $x, t \in \mathbb{R}$. Here $N(u)$ is a differential operator, possibly nonlinear, $P(D)$, $D = d/dx$, is a polynomial operator with constant coefficients, and $a > 0$ is a constant. At the present we make no assumptions and proceed formally. We refer to (1.1) and (1.2) as the species equation and rate equation, respectively. If the kinetics are fast then the rate equation (1.2) becomes the steady state equation

$$P(D)v = u - a,$$

which can be solved by Fourier transforms. Taking the transform on x gives

$$P(-i\xi)\widehat{v} = \widehat{u-a},$$

where ξ is the transform variable. Thus

$$\widehat{v} = H(\xi)\widehat{(u-a)}.$$

1991 *Mathematics Subject Classification.* Primary 35K57; Secondary 92D40.
Key words and phrases. wave fronts, reaction-diffusion equations.

where $H(\xi) = 1/P(-i\xi)$ is the transfer function. Then the impulse response is the function $h = h(x)$ for which $\widehat{h} = H$. Therefore

$$\widehat{v} = \widehat{h}(\xi)\widehat{(u-a)}.$$

By the convolution theorem,

$$v = h * (u - a).$$

Therefore the species equation (1.1) formally becomes

$$u_t = N(u) + u(h * (u - a)), \tag{1.3}$$

which is a nonlinear, nonlocal, differential-integral equation for $u = u(x,t)$ of convolution type.

We are interested in wave front solutions to (1.3), i.e., smooth traveling wave solutions that approach constant states at infinity. Let

$$u = w(s), \quad s = x - ct,$$

where the wave form w and the wave speed c are to be determined, and impose the boundary conditions

$$w(-\infty) = a, \quad w(\infty) = 0. \tag{1.4}$$

Substituting into (1.4) we obtain an ordinary differential equation for w, namely,

$$N(w) + cw' + w(h * (w - a)) = 0, \quad s \in \mathbb{R}. \tag{1.5}$$

Here, *prime* denotes the derivative d/ds. Approximate wave front solutions to (1.5)–(1.4) will yield approximate solution to (1.1)–(1.2). The only assumptions required are $N(0) = 0$, $N(a) = 0$, to ensure that equilibrium states exists at $\pm\infty$, and that the functions involved permit application of Fourier methods.

2. Localization of the Convolution

In many cases the differential operator $P(D)$ will contain coefficients, and therefore the impulse response will contain parameters formed from these constants. In some of these cases the impulse response will have the form

$$h(x) = \lambda h_\eta(x), \tag{2.1}$$

where $\lambda \in \mathbb{R}$ is fixed, η is a positive, real parameter, and the family of density functions h_η satisfies the conditions of a continuous mollifier. That is

$$\begin{array}{ll} \text{(i)} & h_\eta \geq 0 \quad \text{for all} \quad \eta > 0, \\ \text{(ii)} & \int_{-\infty}^{\infty} h_\eta(x)dx = 1 \quad \text{for all} \quad \eta > 0, \\ \text{(iii)} & h_\eta \to \delta \quad as \quad \eta \to 0 \quad \text{(in distribution)}. \end{array} \tag{2.2}$$

Then (1.5) can be written

$$N(w) + cw' - a\lambda w + \lambda w(h_\eta * w) = 0, \quad s \in \mathbb{R}. \tag{2.3}$$

Then we can calculate the convolution by expanding $w(s - y)$ in a Taylor series about s (this is a standard method for reducing nonlocal problems to local ones;

see Murray [5], p. 247ff, p. 487ff; see also Gurley [3]). This yields

$$\begin{aligned}(h_\eta * w)(s) &= \int_{-\infty}^{\infty} h_\eta(y)w(s-y)dy \\ &= \int_{-\infty}^{\infty} h_\eta(y)[w(s) - w'(s)y + \frac{1}{2}w''(s)y^2 + \cdots]dy \\ &= w(s) - M_1 w'(s) + \frac{1}{2}M_2 w''(s) + \cdots + \frac{(-1)^k}{k!}M_k w^{(k)}(s) + \cdots,\end{aligned}$$

where $M_k = \int y^k h_\eta(y)dy$, $k = 1, 2, ...$, represents the kth moment of the density function h_η. Note that M_k depends on the parameter η. Then the wavefront equation (2.3) can be written

(2.4)

$$N(w) + cw' - a\lambda w + \lambda\left(w^2 - M_1 ww' + \frac{1}{2}M_2 ww'' + \cdots + \frac{(-1)^k}{k!}M_k ww^{(k)} + \cdots\right) = 0.$$

In some cases the higher order moments will be small and we can truncate the series to obtain a nonlinear ODE for w whose solution will give approximate wave fronts. In this case the nonlocal effects are localized near the wave front due to the small deviation of the mollifier for small values of the parameter η. The only issue is the existence of an impulse response that satisfies the conditions (2.1)–(2.2).

Therefore, we have a general strategy to obtain approximate wave front solutions to (1.1)–(1.2). We now consider two examples.

3. Example 1

Consider the special case where

$$N(u) = D^2u - \kappa u Du, \quad P(D) = \alpha^2 D^2 - \mu^2,$$

where κ, α, μ are real positive constants. From (1.1)–(1.2) this choice corresponds to the reaction-diffusion system

$$\begin{aligned}u_t &= u_{xx} - \kappa u u_x + uv, \\ \varepsilon v_t &= \alpha^2 v_{xx} - \mu^2 v - u + a.\end{aligned}$$

Here we may interpret u as the population density of an animal species that is diffusing and migrating with a nonlinear migration (convection) speed κu. The quantity v is a decaying, diffusing food source that grows at a rate proportional to $a - u$, for some positive constant a. As we observe below, a can be interpreted as the environment's carrying capacity.

The transfer function associated with the operator $P(D)$ is

$$H(\xi) = -\frac{1}{\alpha^2\xi^2 + \mu^2},$$

and the impulse response is

$$h(x) = -\frac{1}{\mu^2}\frac{1}{2\eta}e^{-|x|/\eta}, \quad \eta = \frac{\alpha}{\mu}.$$

Notice that this has the form (2.1) with $\lambda = -\frac{1}{\mu^2}$ and $h_\eta(x) = \frac{1}{2\eta}e^{-|x|/\eta}$. In this case h_η is even and all the odd moments vanish. Moreover, the even moments can be calculated exactly to obtain (see Gradshteyn and Ryzhik [2], formula 2.321)

$$M_{2k} = \int_{-\infty}^{\infty} y^{2k} \frac{1}{2\eta} e^{-|y|/\eta} dy = \frac{1}{\eta} \int_0^{\infty} y^{2k} e^{-y/\eta} dy = (2k)! \eta^{2k}, \quad k = 1, 2, ...$$

Therefore

$$w'' + (c - \kappa w)w' + \frac{a}{\mu^2} w - \frac{1}{\mu^2}(w^2 + \eta^2 w w'' + \cdots + \eta^{2k} w w^{(2k)} + \cdots) = 0.$$

If $\eta \ll 1$ (i.e., $\alpha \ll \mu$, or the effects of diffusion are small compared to the decay factor), then we expect to have the higher order terms negligible. To leading order we obtain

$$w'' + (c - \kappa w)w' + \frac{a}{\mu^2} w(1 - \frac{w}{a}) = 0. \tag{3.1}$$

Observe that this equation arises from the substitution of assumed wave front solutions $u = w(x - ct)$ into

$$u_t = u_{xx} - \frac{\kappa}{2}(u^2)_x + \frac{a}{\mu^2} u(1 - \frac{u}{a}),$$

which is a reaction-diffusion equation with logistics growth and nonlinear convection; a is the carrying capacity and $\frac{a}{\mu^2}$ is the growth rate. This equation is analyzed in Murray ([5], p. 292ff).

If we keep one additional term, then truncation gives

$$w'' + (c - \kappa w)w' + \frac{a}{\mu^2} w(1 - \frac{w}{a}) - \frac{\eta^2}{\mu^2} w w'' = 0, \quad w(-\infty) = a, \quad w(\infty) = 0. \tag{3.2}$$

We analyze this equation in the next section using the phase plane; the analysis is not too different from that given in Murray [3] for (3.1), and it is only slightly complicated by the presence of the nonlinear term involving w''. A brief summary of phase plane methods is given in Logan [4].

4. Phase Plane Analysis

Equation (3.2) can be reduced to

$$w' = z \tag{4.1}$$

$$z' = -\frac{1}{1 - b^2 w}\left((c - \kappa w)z + \frac{a}{\mu^2} w(1 - \frac{w}{a})\right), \tag{4.2}$$

where $b = \eta/\mu \ll 1$. We take $\kappa = 1$. The only critical points in the (w, z) plane are $(0, 0)$ and $(a, 0)$. See figure 4.1. To show the existence of a wave front solution we must show the existence of a heteroclinic orbit connecting the critical point $(a, 0)$ at $s = -\infty$ to the critical point $(0, 0)$ at $s = +\infty$. Clearly the vector field is to the right for $z > 0$ and to the left if $z < 0$. Along $z = 0$ the vector field is vertical. The horizontal nullcline $z' = 0$ is given by

$$z = \frac{a/\mu^2}{w - c} w \left(1 - \frac{w}{a}\right)$$

if $c \neq a$, and $w = a$, $z = -w/\mu^2$ if $c = a$. Figure 1 shows the case $c > a$. The Jacobi matrix, i.e., the matrix for the linearized system is

$$J(w, z) = \begin{pmatrix} 0 & 1 \\ J_{12} & \frac{w-c}{1-b^2 w} \end{pmatrix},$$

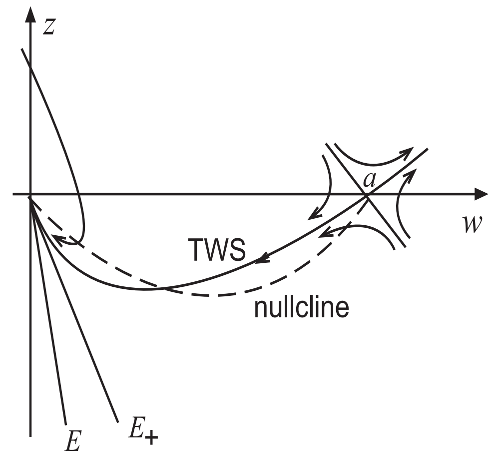

FIGURE 1. wz–phase plane corresponding to the dynamical system (4.1)–(4.2) in the case $c > a$.

where

$$J_{12} = -\frac{(1-b^2w)(-z+a/\mu^2-2w/\mu^2)+b^2((c-w)z+a\mu^{-2}w(1-w/a))}{(1-b^2w)^2}.$$

Then

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -b^2-a/\mu^2 & -c \end{pmatrix}.$$

The eigenvalues are

$$\lambda_\pm = \frac{1}{2}\left(-c \pm \sqrt{c^2-4\left(b^2+a/\mu^2\right)}\right).$$

To ensure that the density w is nonnegative, the critical point $(0,0)$ cannot be a spiral point, an so the discriminant in the last equation must be nonnegative, or

$$c \geq 2\sqrt{b^2+\frac{a}{\mu^2}}. \tag{4.3}$$

This gives a lower bound for the wave speed. The eigenvalues $\lambda_\pm$ in this case are both negative and $(0,0)$ is a stable node. The two eigenvectors (characteristic directions) are $E_\pm = [1, \lambda_\pm]$. These point into the fourth quandrant as shown in figure 4.1, and their slope is steeper than the tangent to the horizontal nullcline $z' = 0$ at the origin. At the critical point $(a,0)$ we have

$$J(a,0) = \begin{pmatrix} 0 & 1 \\ \frac{-b^2+(1-b^2a)a/\mu^2}{(1-b^2a)^2} & \frac{a-c}{1-b^2a} \end{pmatrix}.$$

The eigenvalues in this case are

$$\lambda^*_\pm = \frac{1}{2(1-b^2a)}\left((-(c-a) \pm \sqrt{(c-a)^2-4(b^2-(1-b^2a)a/\mu^2)}\right)$$

and one can show that they are real and have opposite sign for b sufficiently small. Therefore, $(a,0)$ is a saddle point.

Putting all this information together yields the phase diagram shown in figure 1 and a unique heteroclinic orbit of (4.1)–(4.2) connecting $(0,0)$ to $(0,0)$ as shown. This orbit represents a monotonically decreasing wave front of speed c, where c satisfies (4.3).The diagram shows the case when $c > a$. As c decreases there is a bifurcation at $c = a$ where the horizontal nullcline splits into two branches for $c \leq a$. But the wave front exists in these cases as well, provided condition (4.3) holds.

5. Example 2

Now consider an example with nonlinear diffusion given by

$$N(u) = D(uDu), \quad P(D) = \alpha^2 D^2 - \mu^2,$$

with α, μ as in the previous example. From (1.1)–(1.2) this choice corresponds to the reaction-diffusion system

$$u_t = (uu_x)_x + uv, \tag{5.1}$$

$$\varepsilon v_t = \alpha^2 v_{xx} - \mu^2 v - u + a. \tag{5.2}$$

Then the wave front equation (3.1) becomes

$$(ww')' + cw' + rw(1 - \frac{w}{a}) - \frac{\eta^2}{\mu^2} ww'' + \cdots = 0, \quad w(-\infty) = a, \quad w(\infty) = 0,$$

where

$$r = \frac{a}{\mu^2}.$$

The leading order truncation is

$$(ww')' + cw' + rw(1 - \frac{w}{a}) = 0,$$

which arises as the wave front equation for the reaction-diffusion model

$$u_t = (uu_x)_x + ru(1 - \frac{u}{a}).$$

This model has been examined in detail by Aronson [1] (see also Murray [5]). If another term is included in the trunction we have

$$(ww')' + cw' + rw(1 - \frac{w}{a}) - b^2 ww'' = 0, \quad w(-\infty) = a, \quad w(\infty) = 0,$$

where

$$b = \frac{\eta}{\mu}.$$

Letting $z = w'$ we obtain

$$(1 - b^2)wz' = -z^2 - cz - rw(1 - \frac{w}{a}).$$

This equation has a singular point at $w = 0$ and thus we can change independent variables via

$$w\frac{d}{ds} = \frac{d}{d\sigma}.$$

Then we have

$$\frac{dw}{d\sigma} = zw, \tag{5.3}$$

$$(1 - b^2)\frac{dz}{d\sigma} = -z^2 - cz - rw(1 - \frac{w}{a}). \tag{5.4}$$

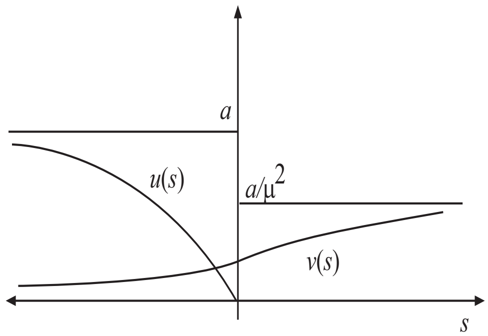

FIGURE 2. Approximate traveling wave solution to the reaction-diffusion system (5.1)–(5.2).

This system, which has three critical points, differs from the model discussed by Aronson [1] by the factor $1-b^2$ appearing in the second equation, so the analysis is similar. Here we examine the effect of this factor on a special, exact wave front connecting the critical points $(a,0)$ and $(0,-c)$ for a special value of c to be determined. The system can be written

$$\frac{dz}{dw} = \frac{-z^2 - cz - rw(1-\frac{w}{a})}{(1-b^2)zw}. \tag{5.5}$$

Now assume a linear relation of the form

$$z = -c_b(1-w/a), \tag{5.6}$$

which connects the two critical points $(a,0)$ and $(0,-c_b)$, where the wave speed c_b is to be determined. Substituting into (5.3) with $c=c_b$ forces

$$c_b = \sqrt{\frac{ra}{2-b^2}} = \sqrt{\frac{ra}{2}}\left(1+\frac{b^2}{4}+\cdots\right).$$

Here, $c_0 = \sqrt{\frac{ra}{2}}$ is the wave speed for the leading order truncated problem. Next, substitute (5.4) into the original equation $dw/ds = z$ to get $dw/ds = -c_b(1-w/a)$. Solving and applying the boundary conditions gives

$$w = a - ae^{c_c(s-s_0)/a},$$

where s_0 is an arbitrary constant. Therefore we have the nonsmooth wave front approximation

$$w(s) = \begin{cases} a - ae^{c_c(s-s_0)/a}, & s \le s_0, \\ 0 & s > s_0. \end{cases} \tag{5.7}$$

The approximation for the density v is

$$\begin{aligned} v(s) &= h*(w-a) = \int_{-\infty}^{\infty} h(s-y)(w(y)-a)dy \\ &= \frac{1}{\mu^2}\left(a - \int_{-\infty}^{\infty} h_\eta(s-y)w(y)dy\right). \end{aligned} \tag{5.8}$$

Plots of the traveling wave approximation (5.5)–(5.6) to (5.1)–(5.2) are shown in figure 2. Note that $v(+\infty) = a/\mu^2$, which is the ratio of the carrying capacity to the decay rate.

6. References

(1) D. G. Aronson, Density-dependent interaction-diffusion systems, in: W.E. Stewart, W.H. Ray, and C.C. Conley, eds., *Dynamics and Modeling of Reactive Systems*, Academic Press, New York (1980).

(2) I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, San Diego (1980).

(3) S.A. Gurley, Traveling front solutions of a nonlocal Fisher equation, *J. Math. Biol.*. 41,272–284 (2000).

(4) J. D. Logan, *Introduction to Nonlinear Partial Differential Equations*, Wiley-Interscience, New York (1994).

(5) J..D. Murray, *Mathematical Biology*, Springer-Verlag, New York, 1993.

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