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Applied Partial Differential Equations, 3rd ed.

Solutions to Selected Exercises

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Contents

Preface

This supplement provides hints, partial solutions, and complete solutions to many of the exercises in Chapters 1 through 5 of *Applied Partial Differential Equations, 3rd edition.*

This manuscript is still in a draft stage, and solutions will be added as the are completed. There may be actual errors and typographical errors in the solutions. I would greatly appreciate any comments or corrections on the manuscript. You can send them to me at:

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J. David Logan

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The Physical Origins of Partial Differential Equations

1.1 Mathematical Models

Exercise 1. Verification that $u = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}$ satisfies the heat equation $u_t = ku_{xx}$ is straightforward differentiation. For larger k, the profiles flatten out much faster.

Exercise 2. The problem is straightforward differentiation. Taking the derivatives is easier if we write the function as $u = \frac{1}{2}\ln(x^2 + y^2)$.

Exercise 5. Integrating $u_{xx} = 0$ with respect to x gives $u_x = \phi(t)$ where ϕ is an arbitrary function. Integrating again gives $u = \phi(t)x + \psi(t)$. But $u(0,t) = \psi(t) = t^2$ and $u(1,t) = \phi(t) \cdot 1 + t^2$, giving $\phi(t) = 1 - t^2$. Thus $u(x,t) = (1-t^2)x + t^2$.

Exercise 6. Leibniz's rule gives

$$u_t = \frac{1}{2}(g(x+ct) + g(x-ct)).$$

Thus

$$u_{tt} = \frac{c}{2}(g'(x+ct) - g'(x-ct)).$$

In a similar manner

$$u_{xx} = \frac{1}{2c}(g'(x+ct) - g'(x-ct)).$$

Thus $u_{tt} = c^2 u_{xx}$.

Exercise 7. If $u = e^{at}\sin bx$ then $u_t = ae^{at}\sin bx$ and $u_{xx} = -b^2 e^{at}\sin bx$. Equating gives $a = -b^2$.

Exercise 8. Letting $v = u_x$ the equation becomes $v_t + 3v = 1$. Multiply by the integrating factor e^{3t} to get

$$\frac{\partial}{\partial t}(ve^{3t}) = e^{3t}.$$

Integrate with respect to t to get

$$v = \frac{1}{3} + \phi(x)e^{-3t},$$

where ϕ is an arbitrary function. Thus

$$u = \int v dx = \frac{1}{3}x + \phi(x)e^{-3t} + \psi(t).$$

Exercise 9. Let $w = e^u$ or $u = \ln w$ Then $u_t = w_t/w$ and $u_x = w_x/w$, giving $w_{xx} = w_{xx}/w - w_x^2/w^2$. Substituting into the PDE for u gives, upon cancelation, $w_t = w_{xx}$.

Exercise 10. It is straightforward to verify that $u = \arctan(y/x)$ satisfies the Laplace equation. We want $u \to 1$ as $y \to 0$ $(x > 0)$, and $u \to -1$ as $y \to 0$ $(x < 0)$. So try

$$u = 1 - \frac{2}{\pi}\arctan\frac{y}{x}.$$

We want the branch of $\arctan z$ with $0 < \arctan z < \pi/2$ for $z > 0$ and $\pi/2 < \arctan z < \pi$ for $z < 0$.

Exercise 11. Differentiate under the integral sign to obtain

$$u_{xx} = \int_0^\infty -\xi^2 c(\xi)e^{-\xi y}\sin(\xi x)d\xi$$

and

$$u_{yy} = \int_0^\infty \xi^2 c(\xi)e^{-\xi y}\sin(\xi x)d\xi.$$

Thus

$$u_{xx} + u_{yy} = 0.$$

Exercise 12b; 13b. Substitute $u = Ae^{i(kx-\omega t)}$ into $u_t + u_{xxx} = 0$ to get the dispersion relation $\omega = -k^3$. The solution is thus

$$u(x,t) = Ae^{i(kx+k^3 t)} = Ae^{ik(x+k^3 t)}.$$

The dispersion relation is real so the PDE is dispersive. Taking the real part we get $u(x,t) = A\cos(k(x+k^2)t)$, which is a left traveling wave moving with speed k^2. Waves with larger wave number move faster.

1.2 Conservation Laws

Exercise 1. Since $A = A(x)$ depends on x, it cannot cancel from the conservation law and we obtain

$$A(x)u_t = -(A(x)\phi)_x + A(x)f.$$

Exercise 2. The solution to the initial value problem is $u(x,t) = e^{-(x-ct)^2}$. When $c = 2$ the wave forms are bell-shaped curves moving to the right at speed 2.

Exercise 3. Letting $\xi = x - ct$ and $\tau = t$, the PDE $u_t + cu_x = -\lambda u$ becomes $U_\tau = -\lambda U$ or $U = \phi(\xi)e^{-\lambda t}$. Thus

$$u(x,t) = \phi(x - ct)e^{-\lambda t}.$$

Exercise 4. In the new dependent variable w the equation becomes $w_t + cw_x = 0$.

Exercise 5. In preparation.

Exercise 7. From Exercise 3 we have the general solution $u(x,t) = \phi(x - ct)e^{-\lambda t}$. For $x > ct$ we apply the initial condition $u(x,0) = 0$ to get $\phi \equiv 0$. Therefore $u(x,t) = 0$ in $x > ct$. For $x < ct$ we apply the boundary condition $u(0,t) = g(t)$ to get $\phi(-ct)e^{-\lambda t} = g(t)$ or $\phi(t) = e^{\lambda t/c}g(-t/c)$. Therefore $u(x,t) = g(t - x/c)e^{-\lambda x/c}$ in $0 \le x < ct$.

Exercise 8. Making the transformation of variables $\xi = x - t$, $\tau = t$, the PDE becomes $U_\tau - 3U = \tau$, where $U = U(\xi, \tau)$. Multiplying through by the integrating factor $\exp(-3\tau)$ and then integrating with respect to τ gives

$$U = -\left(\frac{\tau}{3} + \frac{1}{9}\right) + \phi(\xi)e^{3\tau},$$

or

$$u = -\left(\frac{t}{3} + \frac{1}{9}\right) + \phi(x - t)e^{3t}.$$

Setting $t = 0$ gives $\phi(x) = x^2 + 1/9$. Therefore

$$u = -\left(\frac{t}{3} + \frac{1}{9}\right) + ((x - t)^2 + \frac{1}{9})e^{3t}.$$

Exercise 9. Letting $n = n(x,t)$ denote the concentration in mass per unit volume, we have the flux $\phi = cn$ and so we get the conservation law

$$n_t + cn_x = -r\sqrt{n} \quad 0 < x < l,\ t > 0.$$

The initial condition is $u(x,0) = 0$ and the boundary condition is $u(0,t) = n_0$. To solve the equation go to characteristic coordinates $\xi = x - ct$ and $\tau = t$. Then the PDE for $N = N(\xi, \tau)$ is $N_\tau = -r\sqrt{N}$. Separate variables and integrate to get

$$2\sqrt{N} = -r\tau + \Phi(\xi).$$

Thus

$$2\sqrt{n} = -rt + \Phi(x - ct).$$

Because the state ahead of the leading signal $x = ct$ is zero (no nutrients have arrived) we have $u(x,t) \equiv 0$ for $x > ct$. For $x < ct$, behind the leading signal, we compute Φ from the boundary condition to be $\Phi(t) = 2\sqrt{n_o} - rt/c$. Thus, for $0 < x < ct$ we have

$$2\sqrt{n} = -rt + 2\sqrt{n_0} - \frac{r}{c}(x - ct).$$

Along $x = l$ we have $n = 0$ up until the signal arrives, i.e., for $0 < t < l/c$. For $t > l/c$ we have

$$n(l,t) = (\sqrt{n_0} - \frac{rl}{2c})^2.$$

Exercise 10. The graph of the function $u = G(x + ct)$ is the graph of the function $y = G(x)$ shifted to the *left* ct distance units. Thus, as t increases the profile $G(x+ct)$ moves to the left at speed c. To solve the equation $u_t - cu_x = F(x,t,u)$ on would transform the independent variables via $x = x + ct$, $\tau = t$.

Exercise 11. The conservation law for traffic flow is

$$u_t + \phi_x = 0.$$

If $\phi(u) = \alpha u(\beta - u)$ is chosen as the flux law, then the cars are jammed at the density $u = \beta$, giving no movement or flux; if $u = 0$ there is no flux because there are no cars. The nonlinear PDE is

$$u_t + (\alpha u(\beta - u))_x = 0,$$

or

$$u_t + \alpha(\beta - 2u)u_x = 0.$$

Exercise 12. Transform to characteristic coordinates $\xi = x - vt, \;\; \tau = t$ to get

$$U_\tau = -\frac{\alpha U}{\beta + U}, \quad U = U(\xi, \tau).$$

Separating variables and integrating yields, upon applying the initial condition and simplifying, the implicit equation

$$u - \alpha t - f(x) = \beta \ln(u/f(x)).$$

Graphing the right and left sides of this equation versus u (treating x and $t > 0$ as parameters) shows that there are two crossings, or two roots u; the solution is the smaller of the two.

1.3 Diffusion

Exercise 1. We have$u_{xx}(6,T) \approx (58-2(64)+72)/2^2 = 0.5$. Since $u_t = ku_{xx} > 0$, the temperatue will increase. We have

$$u_t(T,6) \approx \frac{u(T+0.5,6)-u(T,6)}{0.5} \approx ku_{xx}(T,6) \approx 0.02(0.5).$$

This gives $u(T+0.5,6) \approx 64.005$.

Exercise 2. Taking the time derivative

$$E'(t) = \frac{d}{dt}\int_0^l u^2 dx = \int_0^l 2uu_t dx = 2k\int_0^l uu_{xx}dx$$
$$= 2kuu_x \mid_0^l -2k\int u_x^2 \le 0.$$

Thus E in nonincreasing, so $E(t) \le E(0) = \int_0^l u_0(x)dx$. Next, if $u_0 \equiv 0$ then $E(0) = 0$. Therefore $E(t) \ge 0,\ E'(t) \le 0, E(0) = 0$. It follows that $E(t) = 0$. Consequently $u(x,t) = 0$.

Exercise 3. Take

$$w(x,t) = u(x,t) - \frac{h(t)-g(t)}{l}(x-l) - g(t).$$

Then w will satisfy homogeneous boundary conditions. We get the problem

$$w_t = kw_{xx} - F(x,t), \quad 0 < x < l,\ t > 0$$
$$w(0,t) = w(l,t) = 0, \quad t > 0,$$
$$w(x,0) = G(x), \quad 0 < x < l,$$

where

$$F(x,t) = \frac{l-x}{l}(h'(t)-g'(t))-g'(t), \quad G(x) = u(x,0)-\frac{h(0)-g(0)}{l}(x-l)-g(0).$$

Exercise 4. The steady state equation is $u'' - (h/k)u = 0$ which has exponential solution, $u(x) = A\exp(\sqrt{h/k}x) + B\exp(-\sqrt{h/k}x)$. Apply the boundary conditions $u(0) = u(1) = 1$.

Exercise 6. The steady state problem for $u = u(x)$ is

$$ku'' + 1 = 0, \quad u(0) = 0, \ \ u(1) = 1.$$

Solving this boundary value problem by direct integration gives the steady state solution

$$u(x) = -\frac{1}{2k}x^2 = (1 + \frac{1}{2k})x,$$

which is a concave down parabolic temperature distribution.

Exercise 7. The steady-state heat distribution $u = u(x)$ satisfies

$$ku'' - au = 0, \quad u(0) = 1,\ u(1) = 1.$$

The general solution is $u = c_1 \cosh \sqrt{a/k}x + c_2 \sinh \sqrt{a/k}x$. The constants c_1 and c_2 can be determined by the boundary conditions.

Exercise 8. The boundary value problem is

$$\begin{aligned} u_t &= Du_{xx} + ru(1 - u/K), \quad 0 < x < l,\ t > 0, \\ u_x(0,t) &= u_x(l,t) = 0, \quad t > 0, \\ u(x,0) &= ax(l-x), \quad 0 < x < l. \end{aligned}$$

For long times we expect a steady state density $u = u(x)$ to satisfy $-Du'' + ru(1 - u/K) = 0$ with insulated boundary conditions $u'(0) = u'(l) = 0$. There are two obvious solutions to this problem, $u = 0$ and $u = K$. From what we know about the logistics equation

$$\frac{du}{dt} = ru(1 - u/K),$$

(where there is no spatial dependence and no diffusion, and $u = u(t)$), we might expect the the solution to the problem to approach the stable equilibrium $u = K$. In drawing profiles, note that the maximum of the initial condition is $al^2/4$. So the two cases depend on whether this maximum is below the carrying capacity or above it. For example, in the case $al^2/4 < K$ we expect the profiles to approach $u = K$ from below.

Exercise 9. These facts are directly verified using the chain rule to change variables in the equation.

1.4 Diffusion and Randomness

Exercise 1. We have

$$u_t = (D(u)u_x)_x = D(u)u_{xx} + D(u)_x u_x = D(u)u_{xx} + D'(u)u_x u_x.$$

Exercise 2. The steady state equation is $(Du')' = 0$, where $u = u(x)$. If $D =$ constant then $u'' = 0$ which has a linear solution $u(x) = ax + b$. Applying the two end conditions ($u(0) = 4$ and $u'(2) = 1$) gives $b = 4$ and $a = 1$. Thus $u(x) = x + 4$. The left boundary condition means the concentration is held at the value $u = 4$, and the right boundary condition means $-Du'(2) = -D$, meaning that the flux is $-D$. So matter is entering the system at $L = 2$ (moving left). In the second case we have

$$\left(\frac{1}{1+x}u'\right)' = 0.$$

Therefore

$$\frac{1}{1+x}u' = a$$

or

$$u' = a(1+x).$$

The right boundary condition gives $a = 1/3$. Integrating again and applying the left boundary condition gives

$$u(x) = \frac{1}{3}x + \frac{1}{6}x^2 + 4.$$

In the third case the equation is

$$(uu')' = 0,$$

or $uu' = a$.This is the same as

$$\frac{1}{2}(u^2)' = a,$$

which gives

$$\frac{1}{2}u^2 = ax + b.$$

From the left boundary condition $b = 8$. Hence

$$u(x) = \sqrt{2ax + 16}.$$

Now the right boundary condition can be used to obtain the other constant a . Proceeding,

$$u'(2) = \frac{a}{2\sqrt{a+4}} = 1.$$

Thus $a = 2 + \sqrt{20}$.

Exercise 3. The general solution of $Du'' - cu' = 0$ is $u(x) = a + be^{cx/D}$. In the second case the equation is $Du'' - cu' + ru = 0$. The roots of the characteristic polynomial are

$$\lambda_{\pm} = \frac{c}{2D} \pm \frac{\sqrt{c^2 - 4Dr}}{2D}.$$

There are three cases, depending upon upon the discriminant $c^2 - 4Dr$. If $c^2 - 4Dr = 0$ then the roots are equal $(\frac{c}{2D})$ and the general solution has the form

$$u(x) = ae^{cx/2D} + bxe^{cx/2D}.$$

If $c^2 - 4Dr > 0$ then there are two real roots and the general solution is

$$u(x) = ae^{\lambda_+ x} + be^{\lambda_- x}.$$

If $c^2 - 4Dr < 0$ then the roots are complex and the general solution is given by

$$u(x) = ae^{cx/2D}\left(a\cos\frac{\sqrt{4Dr - c^2}}{2D}x + b\sin\frac{\sqrt{4Dr - c^2}}{2D}x\right).$$

Exercise 4. If u is the concentration, use the notation $u = v$ for $0 < x < L/2$, and $u = w$ for $L/2 < x < L$.The PDE model is then

$$v_t = v_{xx} - \lambda v, \quad 0 < x < L/2,$$
$$w_t = w_{xx} - \lambda w, \quad L/2 < x < L.$$

The boundary conditions are clearly $v(0,t) = w(L,t) = 0$, and continuity at the midpoint forces $v(L/2) = w(L/2)$. To get a condition for the flux at the midpoint we take a small interval $[L/2-\epsilon, L/2+\epsilon]$. The flux in at the left minus the flux out at the right must equal 1, the amount of the source. In symbols,

$$-v_x(L/2 - \epsilon, t) + w_x(L/2 + \epsilon) = 1.$$

Taking the limit as $\epsilon \to 0$ gives

$$-v_x(L/2, t) + w_x(L/2) = 1.$$

So, there is a jump in the derivative of the concentration at the point of the source. The steady state system is

$$v'' - \lambda v = 0, \quad 0 < x < L/2,$$
$$w'' - \lambda w = 0, \quad L/2 < x < L,$$

with conditions

$$v(0) = w(L) = 0,$$
$$v(L/2) = w(L/2),$$
$$-v'(L/2) + w'(L/2) = 1.$$

Let $r = \sqrt{\lambda}$. The general solutions to the DEs are

$$v = ae^{rx} + be^{-rx}, \quad w = ce^{rx} + de^{-rx}.$$

The four constants a, b, c, d may be determined by the four auxiliary conditions.

Exercise 5. The steady state equations are

$$v'' = 0, \quad 0 < x < \xi,$$
$$w'' = 0, \quad \xi < x < L,$$

The conditions are

$$v(0) = w(L) = 0,$$
$$v(\xi) = w(\xi),$$
$$-v'(\xi) + w'(\xi) = 1.$$

Use these four conditions to determine the four constants in the general solution to the DEs. We finally obtain the solution

$$v(x) = \frac{\xi - L}{L}x, \quad w(x) = \frac{x - L}{L}\xi.$$

Exercise 6. The equation is

$$u_t = u_{xx} - u_x, \quad 0 < x < L.$$

(With no loss of generality we have taken the constants to be equal to one). Integrating from $x = 0$ to $x = L$ gives

$$\int_0^L u_t dx = \int_0^L u_{xx} dx - \int_0^L u_x dx.$$

Using the fundamental theorem of calculus and bringing out the time derivative gives

$$\frac{\partial}{\partial t}\int_0^L u dx = u_x(L,t) - u_x(0,t) - u(L,t) + u(0,t) = -flux(L,t) + flux(0,t) = 0.$$

Exercise 7. The model is

$$u_t = Du_{xx} + agu_x,$$
$$u(\infty, t) = 0, \quad -Du_x(0,t) - agu(0,t) = 0.$$

The first boundary condition states the concentration is zero at the bottom (a great depth), and the second condition states that the flux through the surface is zero, i.e., no plankton pass through the surface. The steady state equation is

$$Du'' + agu' = 0,$$

which has general solution

$$u(x) = A + Be^{-agx/D}.$$

The condition $u(\infty) = 0$ forces A=0. The boundary condition $-Du'(0) - agu(0) = 0$ is satisfied identically. So we have

$$u(x) = u(0)e^{-agx/D}.$$

Exercise 8. Notice that the dimensions of D are length-squared per unit time, so we use $D = L^2/T$, where L and T are the characteristic length and time, respectively. For sucrose,

$$L = \sqrt{DT} = \sqrt{(4.6 \times 10^{-6})(60 \times 60 \times 24)} = 0.63 \text{ cm}.$$

For the insect,

$$T = \frac{L^2}{D} = \frac{10000^2}{2.0 \times 10^{-1}} = 5 \times 10^8 \text{ sec } = \text{ 6000 days}.$$

Exercise 9. Solve each of the DEs, in linear, cylindrical, and spherical coordinates, respectively:

$$Du'' = 0,$$
$$\frac{D}{r}(ru')' = 0,$$
$$\frac{D}{\rho^2}(\rho^2 u')' = 0.$$

Exercise 10. Let $q = 1 - p$ and begin with the equation

$$u(x, t + \tau) - u(x, t) = pu(x - h, t) + qu(x + h, t) - pu(x, t) - qu(x, t),$$

or

$$u(x, t + \tau) = pu(x - h, t) + qu(x + h, t).$$

Expanding in Taylor series (u and its derivatives are evaluated at (x, t)),

$$u + u_t\tau + \cdots = pu - pu_x h + \frac{1}{2}pu_{xx}h^2 + qu + qu_x h + \frac{1}{2}qu_{xx}h^2 + \cdots,$$

or

$$\tau u_t = (1 - 2p)u_x h + \frac{1}{2}u_{xx}h^2 + \cdots.$$

Then

$$u_t = (1 - 2p)u_x\frac{h}{\tau} + \frac{h^2}{2\tau}u_{xx} + \cdots.$$

Taking the limit as $h, \tau \to 0$ gives

$$u_t = cu_x + Du_{xx},$$

with appropriately defined special limits.

Exercise 11. Similar to the example in the text.

Exercise 12. Draw two concentric circles of radius $r = a$ and $r = b$. The total amount of material in between is

$$2\pi \int_a^b u(r,t) r dr.$$

The flux through the circle $r = a$ is $-2\pi a D u_r(a,t)$ and the flux through r=b is $-2\pi b D u_r(b,t)$. The time rate of change of the total amount of material in between equals the flux in minus the flux out, or

$$2\pi \frac{\partial}{\partial t} \int_a^b u(r,t) r dr. = -2\pi a D u_r(a,t) + 2\pi b D u_r(b,t),$$

or

$$\int_a^b u_t(r,t) r dr = \int_a^b D \frac{\partial}{\partial r}(r u_r(r,t)) dr.$$

Since a and b are arbitrary,

$$u_t(r,t) r = D \frac{\partial}{\partial r}(r u_r(r,t)).$$

1.5 Vibrations and Acoustics

Exercise 1. In balancing the vertical forces, add the term $-\int_0^l g\rho_0(x)dx$ to the right side to account for gravity acting downward.

Exercise 2. In balancing the vertical forces, add the term $-int_0^l \rho_0(x) k u_t dx$ to the right side to account for damping.

Exercise 4. The initial conditions are found by setting $t = 0$ to obtain

$$u_n(x,0) = \sin \frac{n\pi x}{l}.$$

The temporal frequency of the oscillation is $\omega \equiv n\pi c/l$ with period $2\pi/\omega$. As the length l increases, the frequency decreases, making the period of oscillation longer. The tension is τ satisfies $\rho_0 c^2 = \tau$. As τ increases the frequency increases so the oscillations are faster. Thus, tighter strings produce higher frequencies; longer string produce lower frequencies.

Exercise 5. The calculation follows directly by following the hint.

Exercise 6. We have

$$c^2 = \frac{dp}{d\rho} = k\gamma\rho^{\gamma-1} = \frac{\gamma p}{\rho}.$$

Exercise 8. Assume $\tilde{\rho}(x,t) = F(x - ct)$, a right traveling wave, where F is to be determined. Then this satisfies the wave equation automatically and we have $\tilde{\rho}(0,t) = F(-ct) = 1 - 2\cos t$, which gives $F(t) = 1 - 2\cos(-t/c)$. Then

$$\tilde{\rho}(x,t) = 1 - 2\cos(t - x/c).$$

Exercise 9. Differentiate the equations with respect to x and with respect to t. The speed of waves is $\sqrt{1/LC}$.

1.6 Quantum Mechanics

Exercise 1. This is a straightforward verification using rules of differentiation.

Exercise 2. Substitute $y = e^{-ax^2}$ into the Schrödinger equation to get

$$E = \hbar^2 a/m, \quad a^2 = \frac{1}{4}mk/\hbar^2.$$

This gives

$$y(x) = Ce^{-0.5\sqrt{mk}x^2/\hbar}.$$

To find C impose the normality condition $\int_R y(x)^2 dx = 1$ and obtain

$$C = \left(\frac{mk}{2\pi\hbar}\right)^{1/4}.$$

Exercise 4. Let $b^2 \equiv 2mE/\hbar^2$. Then the ODE

$$y'' + b^y = 0$$

has general solution

$$y(x) = A\sin bx + B\cos bx.$$

The condition $y(0) = 0$ forces $B = 0$. The condition $y(\pi) = 0$ forces $\sin b\pi = 0$, and so (assuming $B \neq 0$) b must be an integer, i.e., $n^2 \equiv 2mE/\hbar^2$. The probability density functions are

$$y_n^2(x) = B^2 \sin^2 nx$$

with the constants B chosen such that $\int_0^\pi y^2 dx = 1$. One obtains $B = \sqrt{2/\pi}$. The probabilities are

$$\int_0^0 .25 y_n^2(x) dx.$$

1.7 Heat Flow in Three Dimensions

In these exercises we use the notation ∇ for the gradient operation grad.

Exercise 1. We have

$$div(\nabla u) = \mathrm{div}(u_x, u_y, u_z) = u_{xx} + u_{yy} + u_{zz}.$$

Exercise 2. For nonhomogeneous media the conservation law (1.53) becomes

$$c\rho u_t - \mathrm{div}(K(x, y, z)\nabla u) = f.$$

So the conductivity K cannot be brought out of the divergence.

Exercise 3. Integrate both sides of the PDE over Ω to get

$$\begin{aligned} \int_\Omega f dV &= \int_\Omega -K \Delta u dV = \int_\Omega -K div(\nabla u) dV \\ &= \int_{\partial\Omega} -K \, \nabla u \cdot n dA = \int_{\partial\Omega} g dA. \end{aligned}$$

The left side is the net heat generated inside Ω from sources; the right side is the net heat passing through the boundary. For steady-state conditions, these must balance.

Exercise 4. Follow the suggestion and use the divergence theorem.

Exercise 5. Follow the suggestion in the hint to obtain

$$-\int_\Omega \nabla u \cdot \nabla u \, dV = \lambda \int_\Omega u^2 dV.$$

Both integrals are nonnegative, and so λ must be nonpositive. Note that $\lambda \neq 0$; otherwise $u = 0$.

Exercise 6. This calculation is in the proof of Theorem 4.23 of the text.

Exercise 7. Replace ϕ in equations (1.52) by the given expression and proceed a indicated in the text.

1.8 Laplace's Equation

Exercise 1. The temperature at the origin is the average value of the temperature around the boundary, or

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi}(3\sin 2\theta + 1)d\theta.$$

The maximum and minimum must occur on the boundary. The function $f(\theta) = 3\sin 2\theta + 1$ has an extremum when $f'(\theta) = 0$ or $6\cos 2\theta = 0$. The maxima then occur at $\theta = \pi/4, 5\pi/4$ and the minima occur at $\theta = 3\pi/4, 7\pi/4$.

Exercise 3. We have $u(x) = ax+b$. But $u(0) = b = T_0$ and $u(l) = al+T_0 = T_1$, giving $a = (T_1 - T_0)/l$. Thus

$$u(x) = \frac{T_1 - T_0}{l}x + T_0,$$

which is a straight line connecting the endpoint temperatures. When the right end is insulated the boundary condition becomes $u'(l) = 0$. Now we have $a = 0$ and $b = T_0$ which gives the constant distribution

$$u(x) = T_0.$$

Exercise 5. The boundary value problem is

$$-((1+x^2)u')' = 0, \quad u(0) = 1,\ u(1) = 4.$$

Integrating gives $(1+x^2)u' = c_1$, or $u' = c_1/(1+x^2)$. Integrating again gives

$$u(x) = c_1 \arctan x + c_2.$$

But $u(0) = c_2 = 1$, and $u(1) = c_1 \arctan 1 + 1 = 4$. Then $c_1 = 6/\pi$.

Exercise 6. Assume $u = u(r)$ where $r = \sqrt{x^2+y^2}$. The chain rule gives

$$u_x = u'(r)r_x = u'(r)\frac{x}{\sqrt{x^2+y^2}}.$$

Then, differentiating again using the product rule and the chain rule gives

$$u_{xx} = \frac{x^2}{r^2}u''(r) + \frac{y^2}{r^3}.$$

Similarly

$$u_{yy} = \frac{y^2}{r^2}u''(r) + \frac{x^2}{r^3}.$$

Then

$$\Delta u = u'' + \frac{1}{r}u' = 0.$$

This last equation can be written

$$(ru')' = 0$$

which gives the radial solutions

$$u = a \ln r + b,$$

which are logarithmic. The one dimensional Laplace equation $u'' = 0$ has linear solutions $u = ax + b$, and the three dimensional Laplace equation has algebraic power solutions $u = a\rho^{-1} + b$. In the two dimensional problem we have $u(r) = a \ln r + b$ with $u(1) = 0$ and $u(2) = 10$. Then $b = 0$ and $a = 10/\ln 2$. Thus

$$u(r) = 10\frac{\ln r}{\ln 2}.$$

Exercise 9. We have $\nabla V = E$. Taking the divergence of both sides gives $\Delta V = div\nabla V = div\ E = 0$.

1.9 Classification of PDEs

Exercise 1. The equation

$$u_{xx} + 2ku_{xt} + k^2u_{tt} = 0$$

is parabolic because $B^2 - 4AC = 4k^2 - 4k^2 = 0$. Make the transformation

$$x = \xi, \quad \tau = x - (B/2C)t = x - t/k.$$

Then the PDE reduces to the canonical form $U_{\xi\xi} = 0$. Solve by direct integration. Then $U_\xi = f(\tau)$ and $U = \xi f(\tau) + g(\tau)$. Therefore

$$u = xf(x - t/k) + g(x - t/k),$$

where f, g are arbitrary functions.

Exercise 2. The equation $2u_{xx} - 4u_{xt} + u_x = 0$ is hyperbolic. Make the transformation $\xi = 2x + t, \tau = t$ and the PDE reduces to the canonical form

$$U_{\xi\tau} - \frac{1}{4}U_\xi = 0.$$

Make the substitution $V = U_\xi$ to get $V_\tau = 0.25V$, or $V = F(\xi)e^{\tau/4}$. Then $U = f(\xi)e^{\tau/4} + g(\tau)$, giving

$$u = f(2x + t)e^{t/4} + g(t).$$

Exercise 3. The equation $xu_{xx} - 4u_{xt} = 0$ is hyperbolic. Under the transformation $\xi = t$, $\tau = t + 4\ln x$ the equation reduces to

$$U_{\xi\tau} + \frac{1}{4}U = 0.$$

Proceeding as in Exercise 2 we obtain

$$u = f(t + 4\ln x)e^{t/4} + g(t).$$

Exercise 5. The discriminant for the PDE

$$u_{xx} - 6u_{xy} + 12u_{yy} = 0$$

is $D = -12$ is negative and therefore it is elliptic. Take $b = 1/4 + \sqrt{3}i/12$ and define the complex transformation $\xi = x + by$, $\tau = x + \overline{b}y$. Then take

$$\alpha = \frac{1}{2}(\xi + \tau) = x + \frac{1}{4}y$$

and

$$\beta = \frac{1}{2i}(\xi - \tau) = \frac{\sqrt{3}}{12}y.$$

Then the PDE reduces to Laplace's equation $u_{\alpha\alpha} + u_{\beta\beta} = 0$.

Exercise 6. Change variables as indicated.

Exercise 7a. Hyperbolic when $y < 1/x$, parabolic when $y = 1/x$, elliptic when $y > 1/x$.

2

Partial Differential Equations on Unbounded Domains

2.1 Cauchy Problem for the Heat Equation

Exercise 1a. Making the transformation $r = (x-y)/\sqrt{4kt}$ we have

$$\begin{aligned} u(x,t) &= \int_{-1}^{1} \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt} dy \\ &= -\int_{(x+1)/\sqrt{4kt}}^{(x-1)/\sqrt{4kt}} \frac{1}{\sqrt{\pi}} e^{-r^2} dr \\ &= \frac{1}{2}\left(erf\ \left((x+1)/\sqrt{4kt}\right) - erf\ \left((x-1)/\sqrt{4kt}\right)\right). \end{aligned}$$

Exercise 1b. We have

$$u(x,t) = \int_0^\infty \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt} e^{-y} dy.$$

Now complete the square in the exponent of e and write it as

$$\begin{aligned} -\frac{(x-y)^2}{4kt} - y &= -\frac{x^2 - 2xy + y^2 + 4kty}{4kt} \\ &= -\frac{y + 2kt - x)^2}{4kt} + kt - x. \end{aligned}$$

Then make the substitution in the integral

$$r = \frac{y + 2kt - x}{\sqrt{4kt}}$$

Then

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{(2kt-x)/\sqrt{4kt}}^{\infty} e^{-r^2} dr \\ &= \frac{1}{2} e^{kt-x} \left(1 - erf\left((2kt - x)/\sqrt{4kt}\right)\right). \end{aligned}$$

Exercise 2. We have

$$|u(x,t)| \le \int_R |G(x-y,t)||\phi(y)|dy \le M \int_R G(x-y,t)dy = M.$$

Exercise 3. Use

$$\begin{aligned} erf(z) &= \frac{2}{\sqrt{\pi}} \int_0^z e^{-r^2} dr \\ &= \frac{2}{\sqrt{\pi}} \int_0^z (1 - r^2 + \cdots) dr \\ &= \frac{2}{\sqrt{\pi}} (z - \frac{z^3}{3} + \cdots). \end{aligned}$$

This gives

$$w(x_0, t) = \frac{1}{2} + \frac{x_0}{\pi\sqrt{t}} + \cdots .$$

Exercise 4. The verification is straightforward. We guess the Green's function in two dimensions to be

$$\begin{aligned} g(x,y,t) &= G(x,t)G(y,t) \\ &= \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt} \frac{1}{\sqrt{4\pi kt}} e^{-y^2/4kt} \\ &= \frac{1}{4\pi kt} e^{-(x^2+y^2)/4kt}, \end{aligned}$$

where G is the Green's function in one dimension. Thus g is the temperature distribution caused by a point source at $(x, y) = (0, 0)$ at $t = 0$. This guess gives the correct expression. Then, by superposition, we have the solution

$$u(x,y,t) = \int_{R^2} \frac{1}{4\pi kt} e^{-((x-\xi)^2+(y-\eta)^2)/4kt} \psi(\xi,\eta) d\xi d\eta.$$

Exercise 6. Using the substitution $r = x/\sqrt{4kt}$ we get

$$\int_R G(x,t)dx = \frac{1}{\sqrt{\pi}} \int_R e^{-r^2} dr = 1.$$

Exercise 7. Verification is straightforward. The result does not contradict the theorem because the initial condition is not bounded.

2.2 Cauchy Problem for the Wave Equation

Exercise 1. Applying the initial conditions to the general solution gives the two equations

$$F(x) + G(x) = f(x), \quad -cF'(x) + cG'(x) = g(x).$$

We must solve these to determine the arbitrary functions F and G. Integrate the second equation to get

$$-cF(x) + cG(x) = \int_0^x g(s)ds + C.$$

Now we have two linear equations for F and G that we can solve simultaneously.

Exercise 2. Using d'Alembert's formula we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{2c} \int_{x-ct}^{x+ct} \frac{ds}{1 + 0.25s^2} \\ &= \frac{1}{2c} 2\arctan(s/2) \mid_{x-ct}^{x+ct} \\ &= \frac{1}{c}(\arctan((x+ct)/2) - \arctan((x-ct)/2)). \end{aligned}$$

Exercise 4. Let $u = F(x - ct)$. Then $u_x(0,t) = F'(-ct) = s(t)$. Then

$$F(t) = \int_0^t s(-r/c)dr + K.$$

Then

$$u(x,t) = -\frac{1}{c} \int_0^{t-x/c} s(y)dy + K.$$

Exercise 5. Letting $u = U/\rho$ we have

$$u_{tt} = U_{tt}/\rho, \quad u_\rho = U_\rho/\rho - U/\rho^2$$

and

$$u_{\rho\rho} = U_{\rho\rho}/\rho - 2U_\rho/\rho^2 + 2U/\rho^3.$$

Substituting these quantities into the wave equation gives

$$U_{tt} = c^2 U_{\rho\rho}$$

which is the ordinary wave equation with general solution

$$U(\rho, t) = F(\rho - ct) + G(\rho + ct).$$

Then

$$u(\rho, t) = \frac{1}{r}(F(\rho - ct) + G(\rho + ct)).$$

As a spherical wave propagates outward in space its energy is spread out over a larger volume, and therefore it seems reasonable that its amplitude decreases.

Exercise 6. The exact solution is, by d'Alembert's formula,

$$u(x, t) = \frac{1}{2}(e^{-|x-ct|} + e^{-|x+ct|}) + \frac{1}{2c}(\sin(x + ct) - \sin(x - ct)).$$

Exercise 7. Use the fact that u is has the same value along a characteristic.

Exercise 8. Write

$$v = \int_R H(s, t)u(x, s)ds,$$

where

$$h(s, t) = \frac{1}{\sqrt{4\pi t(k/c^2)}} e^{-s^2/(4t(k/c^2))},$$

which is the heat kernel with k replaced by k/c^2. Thus H satisfies

$$H_t - \frac{k}{c^2} H_{xx} = 0.$$

Then, we have

$$\begin{aligned} v_t - kv_{xx} &= \int_R (H_t(s, t)u(x, s) - kH(s, t)u_{xx}(x, s))ds \\ &= \int_R (H_t(s, t)u(x, s) - (k/c^2)H(s, t)u_{ss}(x, s))ds \end{aligned}$$

where, in the last step, we used the fact that u satisfies the wave equation. Now integrate the second term in the last expression by parts twice. The generated boundary terms will vanish since H and H_s go to zero as $|s| \to \infty$. Then we get

$$v_t - kv_{xx} = \int_R (H_t(s, t)u(x, s) - (k/c^2)H_{ss}(s, t)u(x, s))ds = 0.$$

2.3 Well-Posed Problems

Exercise 1. Consider the two problems

$$u_t + u_{xx} = 0, \quad x \in R,\ t > 0,$$
$$u(x,0) = f(x), \quad x \in R$$

If $f(x) = 1$ the solution is $u(x,t) = 1$. If $f(x) = 1 + n^{-1}\sin nx$, which is a small change in initial data, then the solution is

$$u(x,t) = 1 + \frac{1}{n}e^{n^2 t}\sin nx,$$

which is a large change in the solution. So the solution does not depend continuously on the initial data.

Exercise 2. Integrating twice, the general solution to $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y),$$

where F and G are arbitrary functions. Note that the equation is hyperbolic and therefore we expect the problem to be an evolution problem where data is carried forward from one boundary to another; so a boundary value problem should not be well-posed since the boundary data may be incompatible. To observe this, note that

$$u(x,0) = F(x) + G(0) = f(x). \quad u(x,1) = F(x) + G(1) = g(x),$$

where f and g are data imposed along $y = 0$ and $y = 1$, respectively. But these last equations imply that f and g differ by a constant, which may not be true.

Exercise 3. We subtract the two solutions given by d'Alembert's formula, take the absolute value, and use the triangle inequality to get

$$\begin{aligned}
|u^1 - u^2| &\leq \frac{1}{2}|f^1(x-ct) - f^2(x-ct)| + \frac{1}{2}|f^1(x+ct) - f^2(x+ct)| \\
&\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} |g^1(s) - g^2(s)|ds \\
&\leq \frac{1}{2}\delta_1 + \frac{1}{2}\delta_1 + \frac{1}{2c}\int_{x-ct}^{x+ct} \delta_2 ds \\
&= \delta_1 + \frac{1}{2c}\delta_2(2ct) \\
&\leq \delta_1 + T\delta_2.
\end{aligned}$$

2.4 Semi-Infinite Domains

Exercise 2. We have

$$u(x,t) = \int_0^\infty (G(x-y,t) - G(x+y,t))dy = \text{erf}\,(x/\sqrt{4kt}).$$

Exercise 3. For $x > ct$ we use d'Alembert's formula to get

$$u(x,t) = \frac{1}{2}((x-ct)e^{-(x-ct)} + (x+ct)e^{-(x+ct)}).$$

For $0 < x < ct$ we have from (2.29) in the text

$$u(x,t) = \frac{1}{2}((x+ct)e^{-(x+ct)} - (ct-x)e^{-(ct-x)}).$$

Exercise 4. Letting $w(x,t) = u(x,t) - 1$ we get the problem

$$w_t = kw_{xx}, \quad w(0,t) = 0, \ \ t > 0, \ \ ; w(x,0) = -1, \ \ x > 0.$$

Now we can apply the result of the text to get

$$w(x,t) = \int_0^\infty (G(x-y,t) - G(x+y,t))(-1)dy = -\text{erf}\,(x/\sqrt{4kt}).$$

Then

$$u(x,t) = 1 - erf\ (x/\sqrt{4kt}).$$

Exercise 5. The problem is

$$u_t = ku_{xx}, \ \ x > 0, \ t > 0,$$
$$u(x,0) = 7000, \ \ x > 0,$$
$$u(0,t) = 0, \ \ t > 0.$$

From Exercise 2 we know the temperature is

$$u(x,t) = 7000\ erf\ (x/\sqrt{4kt}) = 7000\frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-r^2}dr.$$

The geothermal gradient at the current time t_c is

$$u_x(0,t_c) = \frac{7000}{\sqrt{\pi k t_c}} = 3.7 \times 10^{-4}.$$

Solving for t gives

$$t_c = 1.624 \times 10^{16} \text{ sec } = 5.15 \times 10^8 \text{ yrs.}$$

This gives a very low estimate; the age of the earth is thought to be about 15 billion years.

There are many ways to estimate the amount of heat lost. One method is as follows. At $t = 0$ the total amount of heat was

$$\int_S \rho cu \, dV = 7000\rho c \frac{4}{3}\pi R^3 = 29321\rho c R^3,$$

where S is the sphere of radius $R = 4000$ miles and density ρ and specific heat c. The amount of heat leaked out can be calculated by integrating the geothermal gradient up to the present day t_c. Thus, the amount leaked out is approximately

$$\begin{aligned}(4\pi R^2)\int_0^{t_c} -Ku_x(0,t)dt &== -4\pi R^2 \rho ck(7000)\int_0^{t_c} \frac{1}{\sqrt{\pi kt}}dt \\ &== -\rho c R^2(1.06 \times 10^{12}).\end{aligned}$$

So the ratio of the heat lost to the total heat is

$$\frac{\rho c R^2(1.06 \times 10^{12})}{29321\rho c R^3} = \frac{3.62 \times 10^7}{R} = 5.6.$$

Exercise 6. Follow the suggested steps. **Exercise 7.** The left side of the equation is the flux through the surface. The first term on the right is Newton's law of cooling and the second term is the radiation heating.

2.5 Sources and Duhamel's Principle

Exercise 1. The solution is

$$\begin{aligned}u(x,t) &= \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \sin s ds \\ &= \frac{1}{c^2}\sin x - \frac{1}{2c^2}(\sin(x-ct) + \sin(x+ct))\end{aligned}$$

Exercise 2. The solution is

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-y, t-\tau) \sin y \, dy d\tau,$$

where G is the heat kernel.

Exercise 3. The problem

$$w_t(x,t,\tau) + cw_x(x,t,\tau) = 0, \quad w(x,0,\tau) = f(x,\tau),$$

has solution (see Chapter 1)

$$w(x,t,\tau) = f(x-ct,\tau).$$

Therefore, by Duhamel's principle, the solution to the original problem is

$$u(x,t) = \int_0^t f(x - c(t-\tau), \tau) d\tau.$$

Applying this formula when $f(x,t) = xe^{-t}$ and $c = 2$ gives

$$u(x,t) = \int_0^t (x - 2(t-\tau))e^{-\tau} d\tau.$$

This integral can be calculated using integration by parts or a computer algebra program. We get

$$u(x,t) = -(x-2t)(e^{-t} - 1) - 2te^{-t} + 2(1 - e^{-t}).$$

2.6 Laplace Transforms

Exercise 4. Using integration by parts, we have

$$\begin{aligned}
L\left(\int_0^t f(\tau)d\tau\right) &= \int_0^\infty \left(\int_0^t f(\tau)d\tau\right) e^{-st} dt \\
&= -\frac{1}{s}\int_0^\infty \left(\int_0^t f(\tau)d\tau\right) \frac{d}{ds}e^{-st} dt \\
&= -\frac{1}{s}\int_0^t f(\tau)d\tau \cdot e^{-st} \,|_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st} dt \\
&= \frac{1}{s}F(s).
\end{aligned}$$

Exercise 5. Since $H = 0$ for $x < a$ we have

$$\begin{aligned} L\left(H(t-a)f(t-a)\right) &= \int_a^\infty f(t-a)e^{-st}dt \\ &= \int_0^\infty f(\tau)e^{-s(\tau+a)}d\tau = e^{-as}F(s). \end{aligned}$$

where we used the substitution $\tau = t - a,\ d\tau = dt$.

Exercise 8. The model is

$$\begin{aligned} u_t &= u_{xx}, \quad x > 0,\ t > 0 \\ u(x,0) &= u_0, \quad x > 0 \\ -u_x(0,t) &= -u(0,t). \end{aligned}$$

Taking the Laplace transform of the PDE we get

$$U_{xx} - sU = -u_0.$$

The bounded solution is

$$U(x,s) = a(s)e^{-x\sqrt{s}} + \frac{u_0}{s}.$$

The radiation boundary condition gives

$$-a(s)\sqrt{s} = a(s) + \frac{u_0}{s}$$

or

$$a(s) = -\frac{u_0}{s(1+\sqrt{s})}.$$

Therefore, in the transform domain

$$U(x,s) = -\frac{u_0}{s(1+\sqrt{s})}e^{-x\sqrt{s}} + \frac{u_0}{s}.$$

Using a table of Laplace transforms we find

$$u(x,t) = u_0 - u_0\left[\operatorname{erfc}\left(\frac{x}{\sqrt{4t}}\right) - \operatorname{erfc}\left(\sqrt{t} + \frac{x}{\sqrt{4t}}\right)e^{x+t}\right].$$

where $\operatorname{erfc}(z) = 1 - \operatorname{erf}(z)$.

Exercise 10. Taking the Laplace transform of the PDE gives, using the initial conditions,

$$U_{xx} - \frac{s^2}{c^2}U = -\frac{g}{sc^2}.$$

The general solution is

$$U(x,s) = A(s)e^{-sx/c} + B(s)e^{sx/c} + \frac{g}{s^3}.$$

To maintain boundedness, set $B(s) = 0$. Now $U(0,s) = 0$ gives $A(s) = -g/s^3$. Thus

$$U(x,s) = -\frac{g}{s^3}e^{-sx/c} + \frac{g}{s^3}$$

is the solution in the transform domain. Now, from a table or computer algebra program,

$$L^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2}, \quad L^{-1}(F(s)e^{-as}) = H(t-a)f(t-a).$$

Therefore

$$L^{-1}\left(\frac{1}{s^3}e^{-xs/c}\right) = H(t-x/c)\frac{(t-x/c)^2}{2}.$$

Hence

$$u(x,t) = \frac{gt^2}{2} - gH(t-x/c)\frac{(t-x/c)^2}{2}.$$

Exercise 11. Taking the Laplace transform of the PDE while using the initial condition gives, for $U = U(x,y,s)$,

$$U_{yy} - pU = 0.$$

The bounded solution of this equation is

$$U = a(x,s)e^{-y\sqrt{s}}.$$

The boundary condition at $y = 0$ gives $sU(x,o,s) = -U_x(x,0,s)$ or $a = -a_x$, or

$$a(x,s) = f(s)e^{-xs}.$$

The boundary condition at $x = u = 0$ forces $f(s) = 1/s$. Therefore

$$U(x,y,s) = \frac{1}{s}e^{-xs}e^{-y\sqrt{s}}.$$

From the table of transforms

$$u(x,y,t) = 1 - \text{erf}((y-x)/\sqrt{4t}).$$

Exercise 13. Taking the Laplace transform of the PDE gives, using the initial conditions,

$$U_{xx} - \frac{s^2}{c^2}U = 0.$$

The general solution is

$$U(x,s) = A(s)e^{-sx/c} + B(s)e^{sx/c}.$$

To maintain boundedness, set $B(s) = 0$. Now The boundary condition at $x = 0$ gives $U(0,s) = G(s)$ which forces $A(s) = G(s)$. Thus

$$U(x,s) = G(s)e^{-sx/c}.$$

Therefore, using Exercise 4, we get

$$u(x,t) = H(t - x/c)g(t - x/c).$$

Exercise 14. The problem is

$$\begin{aligned} u_t &= u_{xx}, \quad x,t > 0 \\ u(x,0) &= 0, \quad x > 0 \\ u(0,t) &= f(t), \quad t > 0. \end{aligned}$$

Taking Laplace transforms and solving gives

$$U(x,s) = F(s)e^{-x\sqrt{s}}$$

Here we have discarded the unbounded part of the solution. So, by convolution,

$$u(x,t) = \int_0^t f(\tau)\frac{x}{\sqrt{4\pi(t-\tau)^3}}e^{-x^2/4(t-\tau)}d\tau.$$

Hence, evaluating at $x = 1$,

$$U(t) = \int_0^t f(\tau)\frac{1}{\sqrt{4\pi(t-\tau)^3}}e^{-1/4(t-\tau)}d\tau,$$

which is an integral equation for $f(t)$. Suppose $f(t) = f_0$ is constant and $U(5) = 10$. Then

$$\frac{20\sqrt{\pi}}{f_0} = \int_0^5 \frac{e^{-1/4(5-\tau)}}{\sqrt{4\pi(5-\tau)^3}}d\tau = 2\ erfc\ (1/\sqrt{5}),$$

where erfc $= 1 -$ erf. Thus $f_0 = 59.9$ degrees.

2.7 Fourier Transforms

Exercise 1. The convolution is calculated from

$$x \star e^{-x^2} = \int_{-\infty}^{\infty} (y - x)e^{-y^2}\,dy.$$

Exercise 2. From the definition we have

$$\begin{aligned}
\mathcal{F}^{-1}(e^{-a|\xi|})) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a|\xi|}e^{-ix\xi}d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{a\xi}e^{-ix\xi}d\xi + \frac{1}{2\pi}\int_{0}^{\infty} e^{-a\xi}e^{-ix\xi}d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{a\xi-ix\xi}d\xi + \frac{1}{2\pi}\int_{0}^{\infty} e^{-a\xi-ix\xi}d\xi \\
&= \frac{1}{2\pi}\frac{1}{a-ix}e^{(a-ix)\xi}\;|_{-\infty}^{0} + \frac{1}{2\pi}\frac{1}{-a-ix}e^{(-a-ix)\xi}\;|_{0}^{\infty} \\
&= \frac{a}{\pi}\frac{1}{a^2+x^2}.
\end{aligned}$$

Exercise 3a. Using the definition of the Fourier transform

$$2\pi\mathcal{F}^{-1}(-\xi) = \int_{-\infty}^{\infty} u(x)e^{-i(-\xi)x}dx = \mathcal{F}(u)(\xi).$$

Exercise 3b. From the definition,

$$\begin{aligned}
\hat{u}(\xi + a) &= \int_{-\infty}^{\infty} u(x)e^{i(\xi+a)x}dx \\
&= \int_{-\infty}^{\infty} u(x)e^{iax}e^{i\xi x}dx \\
&= \mathcal{F}(e^{iax}u)(\xi).
\end{aligned}$$

Exercise 3c. Use 3(a) or, from the definition,

$$\mathcal{F}(u(x+a)) = \int_{-\infty}^{\infty} u(x+a)e^{i\xi x}dx = \int_{-\infty}^{\infty} u(y)e^{i\xi(y-a)}dy = e^{-ia\xi}\hat{u}(\xi).$$

Exercise 6. From the definition

$$\begin{aligned}\hat{u}(\xi) &= \int_0^\infty e^{-ax}e^{i\xi x}dx \\ &= \int_0^\infty e^{(i\xi-a)x}dx \\ &= \frac{1}{i\xi-a}e^{(i\xi-a)x}\ |_0^\infty \\ &= \frac{1}{a-i\xi}.\end{aligned}$$

Exercise 7. Observe that

$$xe^{-ax^2} = -\frac{1}{2a}\frac{d}{dx}e^{-ax^2}.$$

Then

$$\mathcal{F}(xe^{-ax^2}) = -\frac{1}{2a}(-i\xi)\mathcal{F}(e^{-ax^2}).$$

Exercise 9. Take transforms of the PDE to get

$$\hat{u}_t = (-i\xi)^2\hat{u} + \hat{f}(\xi,t).$$

Solving this as a linear, first order ODE in t with ξ as a parameter, we get

$$\hat{u}(\xi,t) = \int_0^t e^{-x^2(t-\tau)}\hat{f}(\xi,\tau)d\tau.$$

Taking the inverse Fourier transform, interchanging the order of integration, and applying the convolution theorem gives

$$\begin{aligned}u(x,t) &= \int_0^t \mathcal{F}^{-1}\left[e^{-x^2(t-\tau)}\hat{f}(\xi,\tau)\right]d\tau \\ &= \int_0^t \mathcal{F}^{-1}\left[e^{-x^2(t-\tau)}\right]\star f(x,\tau)d\tau \\ &= \int_0^t\int_{-\infty}^\infty \frac{1}{\sqrt{4\pi(t-\tau)}}e^{-(x-y)^2/4(t-\tau)}f(y,\tau)dyd\tau.\end{aligned}$$

Exercise 10. Proceeding exactly in the same way as in the derivation of (2.65) in the text, but with k replaced by I, we obtain the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi it}}\int_{-\infty}^\infty e^{-(x-y)^2/4it}f(y)dy,$$

where $u(x,0)=f(x)$. Thus

$$u(x,t)=\frac{1}{\sqrt{4\pi it}}\int_{-\infty}^{\infty}e^{i(x-y)^2/4t-y^2}dy.$$

Here, in the denominator, $\sqrt{i}$ denotes the root with the positive real part, that is $\sqrt{i}=(1+i)/\sqrt{2}$.

Exercise 11. Letting $v=u_y$ we have

$$v_{xx}+v_{yy}=0,\quad x\in R,\ y>0;\quad v(x,0)=g(x).$$

Hence

$$v(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yg(\tau)}{(x-\tau)^2+y^2}d\tau.$$

Then

$$\begin{aligned}
u(x,y)&=\int_0^y v(x,\xi)d\xi\\
&=\int_0^y\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yg(\tau)}{(x-\tau)^2+y^2}d\tau d\xi\\
&=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^y\frac{y}{(x-\tau)^2+y^2}d\xi d\tau\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}g(\tau)\left(\ln\left((x-\tau)^2+y^2\right)-\ln\left((x-\tau)^2\right)\right)d\tau\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}g(x-\xi)\ln\left(\xi^2+y^2\right)d\tau+C.
\end{aligned}$$

Exercise 12. We have

$$\begin{aligned}
u(x,y)&=\frac{y}{\pi}\int_{-l}^{l}\frac{d\tau}{(x-\tau)^2+y^2}\\
&=\frac{1}{\pi}\left(\arctan\left(\frac{l-x}{y}\right)+\arctan\left(\frac{l+x}{y}\right)\right).
\end{aligned}$$

Exercise 13. From the definition of the Fourier transform

$$\begin{aligned}
\mathcal{F}(u_x)&=\int_{-\infty}^{\infty}u_x(x,t)e^{i\xi x}dx\\
&=u(x,t)e^{i\xi x}\mid_{-\infty}^{\infty}-i\xi\int_{-\infty}^{\infty}u(\xi,t)e^{i\xi x}dx\\
&=-i\xi\hat{u}(\xi,t).
\end{aligned}$$

For the second derivative, integrate by parts twice and assume u and u_x tend to zero as $x \to \pm\infty$ to get rid of the boundary terms.

Exercise 14. In this case where f is a square wave signal,

$$\mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx = \int_{-a}^{a} e^{i\xi x} dx = \frac{2\sin \xi x}{\xi}.$$

Exercise 15. Taking the Fourier transform of the PDE

$$u_t = Du_{xx} - cu_x$$

gives

$$\hat{u}_t = -(D\xi^2 + i\xi c)\hat{u},$$

which has general solution

$$\hat{u}(\xi, t) = C(\xi) e^{-D\xi^2 t - i\xi ct}.$$

The initial condition forces $C(\xi) = \hat{\phi}(\xi)$ which gives

$$\hat{u}(\xi, t) = \hat{\phi}(\xi) e^{-D\xi^2 t - i\xi ct}.$$

Using

$$\mathcal{F}^{-1}\left(e^{-D\xi^2 t}\right) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}$$

and

$$\mathcal{F}^{-1}\left(\hat{u}(\xi, t) e^{-ia\xi}\right) = u(x + a),$$

we have

$$\mathcal{F}^{-1}\left(e^{-i\xi ct} e^{-D\xi^2 t}\right) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x+vt)^2/4Dt}.$$

Then, by convolution,

$$u(x, t) = \phi \star \frac{1}{\sqrt{4\pi Dt}} e^{-(x+vt)^2/4Dt}.$$

Exercise 16. (a) Substituting $u = \exp(i(kx - \omega t))$ into the PDE $u_t + u_{xxx} = 0$ gives $-i\omega + (ik)^3 = 0$ or $\omega = -k^3$. Thus we have solutions of the form

$$u(x, t) = e^{i(kx + k^3 t)} = e^{ik(x + k^2 t)}.$$

The real part of a complex-valued solution is a real solution, so we have solutions of the form

$$u(x, t) = \cos[k(x + k^2 t)].$$

These are left traveling waves moving with speed k^2. So the temporal frequency ω as well as the wave speed $c = k^2$ depends on the spatial frequency, or wave number, k. Note that the wave length is proportional to $1/k$. Thus, higher frequency waves are propagated faster.

(b) Taking the Fourier transform of the PDE gives

$$\hat{u}_t = -(-i\xi)^3\hat{u}.$$

This has solution

$$\hat{u}(\xi, t) = \hat{\phi}(\xi)e^{-i\xi^3 t},$$

where $\hat{\phi}$ is the transform of the initial data. By the convolution theorem,

$$u(x,t) = \phi(x) \star F^{-1}(e^{-i\xi^3 t}).$$

To invert this transform we go to the definition of the inverse. We have

$$\begin{aligned}
\mathcal{F}^{-1}(e^{-i\xi^3 t}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi^3 t}e^{-i\xi x}d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(\xi^3 t + \xi x)d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\left(\frac{z^3}{3} + \frac{zx}{(3t)^{1/3}}\right)\frac{1}{(3t)^{1/3}}dz \\
&= \frac{1}{(3t)^{1/3}}\mathrm{Ai}\left(\frac{x}{(3t)^{1/3}}\right).
\end{aligned}$$

where we made the substitution $\xi = z/(3t)^{1/3}$ to put the integrand in the form of that in the Airy function. Consequently we have

$$u(x,t) = \frac{1}{(3t)^{1/3}}\int_{-\infty}^{\infty} \phi(x-y)\mathrm{Ai}\left(\frac{y}{(3t)^{1/3}}\right)dy.$$

Exercise 18. The problem is

$$\begin{gathered}
u_{tt} = c^2 u_{xx} = 0, \quad x \in R,\ t > 0, \\
u(x,0) = f(x),\ \ u_t(x,0) = 0,\ \ x \in R.
\end{gathered}$$

Taking Fourier transforms of the PDE yields

$$\hat{u}_{tt} + c^2\xi^2\hat{u} = 0$$

whose general solution is

$$\hat{u} = A(\xi)e^{i\xi ct} + B(\xi)e^{-i\xi ct}.$$

From the initial conditions, $\hat{u}(\xi, 0) = \hat{f}(\xi)$ and $\hat{u}_t(\xi, 0) = 0$. Thus $A(\xi) = B(\xi) = 0.5\hat{f}(\xi)$. Therefore

$$\hat{u}(\xi, t) = 0.5\hat{f}(\xi)(e^{i\xi ct} + e^{-i\xi ct}).$$

Now we use the fact that

$$F^{-1}\left(\hat{f}(\xi)e^{ia\xi}\right) = f(x-a)$$

to invert each term. Whence

$$u(x,t) = 0.5(f(x-ct) + f(x+ct))$$

Exercise 19. The problem

$$u_t = Du_{xx} - vu_x, \quad x \in R,\ t > 0; \quad u(x,0) = e^{-ax^2}, \ \ x \in R$$

can be solved by Fourier transforms to get

$$u(x,t) = \frac{\sqrt{a}}{\sqrt{a+4Dt}} e^{-(x-vt)^2/((a+4Dt)}.$$

Thus, choosing $a = v = 1$ we get

$$U(t) = u(1,t) = \frac{1}{\sqrt{1+4Dt}} e^{-(1-t)^2/((1+4Dt)}.$$

Exercise 20. Notice the left side is a convolution. Take the transform of both sides, use the convolution theorem, and solve for $\hat{f}$. Then invert to get f.

3
Orthogonal Expansions

3.1 The Fourier Method

Exercise 1. Form the linear combination

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos nct \sin nx.$$

Then

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin nx.$$

Using the exactly same calculation as in the text, we obtain

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \; dx.$$

Observe that $u_t(x,0) = 0$ is automatically satisfied.

When the initial conditions are changed to $u(x,0) = 0,\ u_t(x,0) = g(x)$ then a linear combination of the fundamental solutions $u_n(x,t) = \cos nct \sin nx$ does not suffice. But, observe that $u_n(x,t) = \sin nct \sin nx$ now works and form the linear combination

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nct \sin nx.$$

Now $u(x,0) = 0$ is automatically satisfied and

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} ncb_n \sin nx.$$

Again using the argument in (3.5)–(3.7), one easily shows the b_n are given by

$$b_n = \frac{2}{nc\pi}\int_0^\pi g(x)\sin nx\, dx.$$

3.2 Orthogonal Expansions

Exercise 2. The requirement for orthogonality is

$$\int_0^\pi \cos mx \cos nx\, dx = 0, \quad m \neq n.$$

For the next part make the substitution $y = \pi x/l$ to get

$$\int_0^l \cos(m\pi x/l)\cos(n\pi x/l); dx = \int_0^\pi \cos my \cos ny\, dy = 0, \quad m \neq n.$$

We have

$$c_n = \frac{(f, \cos(n\pi x/l))}{||\cos(n\pi x/l)||^2}.$$

Thus

$$c_0 = \frac{1}{l}\int_0^l f(x)dx, \quad c_n = \frac{2}{l}\int_0^l f(x)\cos(n\pi x/l)dx, \;\; n \geq 1.$$

Exercise 8. Up to a constant factor, the Legendre polynomials are

$$P_0(x) = 1,\; P_1(x) = x,\; P_2(x) = \frac{3}{2}x^2 - \frac{1}{2},\; P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x.$$

The coefficients c_n in the expansion are given by the generalized Fourier coefficients

$$c_n = \frac{1}{||P_n||^2}(f, P_n) = \frac{\int_{-1}^1 e^x P_n(x)dx}{\int_{-1}^1 Pn(x)^2 dx}.$$

The pointwise error is

$$E(x) = e^x - \sum_{n=0}^3 c_n P_n(x).$$

The mean square error is

$$E = \left(\int_{-1}^1 [e^x - \sum_{n=0}^3 c_n P_n(x)]^2 dx\right)^{1/2}.$$

The maximum pointwise error is $\max_{-1\leq x\leq 1}|E(x)|$.

Exercise 9. Expanding

$$\begin{aligned} q(t) &= (f + tg, f + tg) \\ &= ||g||^2 t^2 + 2(f, g)t + ||f||^2. \end{aligned}$$

which is a quadratic in t. Because $q(t)$ is nonnegative (a scalar product of a function with itself is necessarily nonnegative because it is the norm-squared), the graph of the quadratic can never dip below the t axis. Thus it can have at most one real root. Thus the discriminant $b^2 - 4ac$ must be nonpositive. In this case the discriminant is

$$b^2 - 4ac = 4(f, g)^2 - 4||g||^2||f||^2 \leq 0.$$

This gives the desired inequality.

Exercise 12. Use the calculus facts that

$$\int_0^b \frac{1}{x^p} dx < \infty, \quad p < 1$$

and

$$\int_a^\infty \frac{1}{x^p} dx < \infty, \quad p > 1 \ \ (a > 0).$$

Otherwise the improper integrals diverge. Thus $x^r \in L^2[0, 1]$ if $r > -1/2$ and $x^r \in L^2[0, \infty]$ if $r < -1/2$ *and* $r > -1/2$, which is impossible.

Exercise 13. We have

$$\cos x = \sum_{n=1}^{\infty} b_n \sin 2nx, \quad b_n = \frac{4}{\pi} \int_0^{\pi/2} \cos x \sin 2nx \, dx.$$

Also,

$$\sin x = \sum_{n=1}^{\infty} b_n \sin nx$$

clearly forces $b_1 = 1$ and $b_n = 0$ for $n \geq 1$. Therefore the Fourier series of $\sin x$ on $[0, \pi]$ is just a single term, $\sin x$.

3.3 Classical Fourier Series

Exercise 1. Since f is an even function, $b_n = 0$ for all n. We have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} dx = 1,$$

and for $n = 1, 2, 3, \ldots,$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nx \, dx = \frac{2}{n\pi}\sin(n\pi/2)$$

Thus the Fourier series is

$$\begin{aligned} &\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi/2)\cos nx \\ &= \frac{1}{2} + \frac{2}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \cdots\right). \end{aligned}$$

A plot of a two-term and a four-term approximation is shown in the figure.

Exercise 3. Because the function is even, $b_n = 0$. Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = 2\pi^2/3$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{4(-1)^n}{n^2}.$$

So the Fourier series is

$$\frac{\pi^2}{3} + \sum_{n=0}^{\infty}\frac{4(-1)^n}{n^2}\cos nx.$$

This series expansion of $f(x) = x^2$ must converge to $f(0) = 0$ at $x = 0$ since f is piecewise smooth and continuous there. This gives

$$0 = \frac{\pi^2}{3} + \sum_{n=0}^{\infty}\frac{4(-1)^n}{n^2},$$

which implies the result.

The frequency spectrum is

$$\gamma_0 = \frac{2\pi^2}{3\sqrt{2}}, \quad \gamma_n = \frac{4}{n^2}, \; n \geq 1.$$

Exercise 7. This problem suits itself for a computer algebra program to calculate the integrals. We find $a_0 = 1$ and $a_n = 0$ for $n \geq 1$. Then we find

$$\begin{aligned} b_n &= \frac{1}{2\pi}\int_{-2\pi}^{0}(x+1)\sin(nx/2)dx + \frac{1}{2\pi}\int_{0}^{2\pi} x\sin(nx/2)dx \\ &= \frac{1}{\pi}\frac{-1 + (-1)^n + 4\pi(-1)^{n+1}}{n}. \end{aligned}$$

Note that $b_n = -4/n\pi$ if n is even and $b_n = (-2+4\pi)/n\pi$ if n is odd. So the Fourier series is

$$f(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{-1+(-1)^n+4\pi(-1)^{n+1}}{n}\sin(nx/2).$$

A five-term approximation is shown in the figure.

Exercise 10. Because $\cos ax$ is even we have $b_n = 0$ for all n. Next

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos ax\; dx = \frac{2\sin a\pi}{a\pi}$$

and, using a table of integrals or a software program, for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos ax \cos nx\; dx \\ &= \frac{1}{\pi}\left(\frac{\sin(a-n)x}{2(a-n)} + \frac{\sin(a+n)x}{2(a+n)}\right)_{-\pi}^{\pi} \\ &= \frac{2a(-1)^n}{\pi(a^2-n^2)}\sin a\pi. \end{aligned}$$

Therefore the Fourier series is

$$\cos ax = \frac{\sin a\pi}{a\pi} + \sum_{n=1}^{\infty} \frac{2a(-1)^n}{\pi(a^2-n^2)}\sin a\pi \cos nx.$$

Substitute $x = 0$ to get the series for $\csc a\pi$.

Exercise 11. Here $f(x)$ is odd so $a_n = 0$ for all n. Then

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{0} -\frac{1}{2}\sin nx\; dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{1}{2}\sin nx\; dx \\ &= \frac{1}{n\pi}(1-(-1)^n). \end{aligned}$$

Therefore

$$b_= \frac{2}{(2k-1)\pi}, \quad k = 1, 2, 3, \ldots.$$

The Fourier series is

$$\sum_{k=1} \infty \frac{2}{(2k-1)\pi}\sin(2k-1)x.$$

Plots of partial sums show overshoot near the discontinuity (Gibbs phenomenon).

4
Partial Differential Equations on Bounded Domains

4.1 Overview of Separation of Variables

Exercise 2. The solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin nx,$$

where

$$a_n = \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin nx dx = -\frac{2}{n\pi}((-1)^n - \cos(n\pi/2)).$$

Thus

$$u(x,t) = \frac{2}{\pi}e^{-t}\sin x - \frac{2}{\pi}e^{-4t}\sin 2x + \frac{2}{3\pi}e^{-9t}\sin 3x + \frac{2}{5\pi}e^{-25t}\sin 5x + \cdots .$$

Exercise 3. The solution is given by formula (4.14) in the text, where the coefficients are given by (4.15) and (4.16). Since $G(x) = 0$ we have $c_n = 0$. Then

$$d_n = \frac{2}{\pi}\int_0^{\pi/2} x \sin nx \, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin nx \, dx.$$

Using the antiderivative formula $\int x \sin nx \, dx = (1/n^2)\sin nx - (x/n)\cos nx$ we integrate to get

$$d_n = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}.$$

Exercise 4. Substituting $u = y(x)g(t)$ into the PDE and boundary conditions gives the SLP

$$-y'' = \lambda y, \quad y(0) = y(1) = 0$$

and, for g, the equation

$$g'' + kg' + c^2\lambda g = 0.$$

The SLP has eigenvalues and eigenfunctions

$$\lambda_n = n^2\pi^2, \quad y_n(x) = \sin n\pi x, \;\; n = 1, 2, \ldots.$$

The g equation is a linear equation with constant coefficients; the characteristic equations is

$$m^2 + km + c^2\lambda = 0,$$

which has roots

$$m = \frac{1}{2}(-k \pm \sqrt{k^2 - 4c^2n^2\pi^2}).$$

By assumption $k < 2\pi c$, and therefore the roots are complex for all n. Thus the solution to the equation is (see the Appendix in the text on ordinary differential equations)

$$g_n(t) = e^{-kt}(a_n \cos(m_n t) + b_n \sin(m_n t)),$$

where

$$m_n = \frac{1}{2}\sqrt{4c^2n^2\pi^2 - k^2}).$$

Then we form the linear combination

$$u(x,t) = \sum_{n=1}^{\infty} e^{-kt}(a_n \cos(m_n t) + b_n \sin(m_n t)) \sin(n\pi x).$$

Now apply the initial conditions. We have

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x),$$

and thus

$$a_n = \int_0^1 f(x) \sin n\pi x \; dx.$$

The initial condition $u_t = 0$ at $t = 0$ yields

$$u_t(x,0) = 0 = \sum_{n=1}^{\infty} (b_n m_n - k a_n) \sin(n\pi x).$$

Therefore

$$b_n m_n - k a_n = 0,$$

or

$$b_n = \frac{k a_n}{m_n} = \frac{k}{m_n} \int_0^1 f(x) \sin n\pi x \; dx.$$

4.2 Sturm–Liouville Problems

Exercise 1. Substituting $u(x,t) = g(t)y(x)$ into the PDE

$$u_t = (p(x)u_x)_x - q(x)u$$

gives

$$g'(t)y(x) = \frac{d}{dx}(p(x)g(t)y'(x)) - q(x)g(t)y(x).$$

Dividing by $g(t)y(x)$ gives

$$\frac{g'}{g} = \frac{(py')' - q}{y}.$$

Setting these equal to $-\lambda$ gives the two differential equations for g and y.

Exercise 2. When $\lambda = 0$ the ODE is $-y'' = 0$ which gives $y(x) = ax + b$. But $y'(0) = a = 0$ and $y(l) = al + b = 0$, and so $a = b = 0$ and so zero is not an eigenvalue. When $\lambda = -k^2 < 0$ then the ODE has general solution $y(x) = ae^{kx} + be^{-kx}$, which are exponentials. If $y = 0$ at $x = 0$ and $x = l$, then it is not difficult to show $a = b = 0$, which means that there are no negative eigenvalues. If $\lambda = k^2 > 0$ then $y(x) = a\sin kx + b\cos kx$. Then $y'(0) = 0$ forces $a = 0$ and then $y(l) = b\cos kl = 0$. But the cosine function vanishes at $\pi/2$ plus a multiple of π, i.e.,

$$kl = \sqrt{\lambda}l = \pi/2 + n\pi$$

for $n = 0, 1, 2, \ldots$. This gives the desired eigenvalues and eigenfunctions as stated in the problem.

Exercise 3. The problem is

$$-y'' = \lambda y, \quad y(0) + y'(0) = 0, \; y(1) = 0.$$

If $\lambda = 0$ then $y(x) = ax + b$ and the boundary conditions force $b = -a$. Thus eigenfunctions are

$$y(x) = a(1 - x).$$

If $\lambda < 0$ then $y(x) = a\cosh kx + b\sinh kx$ where $\lambda = -k^2$. The boundary conditions give

$$a + bk = 0, \quad a\cosh k + b\sinh k = 0.$$

Thus $\sinh k - k\cosh k = 0$ or $k = \tanh k$ which has no nonzero roots. Thus there are no negative eigenvalues.

If $\lambda = k^2 > 0$ then $y(x) = a\cos kx + b\sin kx$. The boundary conditions imply

$$a + bk = 0, \quad a\cos k + b\sin k = 0.$$

Thus $k = \tan k$ which has infinity many positive roots k_n (note that the graphs of k and $\tan k$ cross infinitely many times). So there are infinitely many positive eigenvalues given by $\lambda_n = k_n^2$.

Exercise 4. The SLP is

$$-y'' = \lambda y, \quad y(0) + 2y'(0) = 0,\ 3y(2) + 2y'(2) = 0.$$

If $\lambda = 0$ then $y(x) = ax + b$. The boundary conditions give $b + 2a = 0$ and $8a + 3b = 0$ which imply $a = b = 0$. So zero is not an eigenvalue. Since this problem is a regular SLP we know by the fundamental theorem that there are infinitely many positive eigenvalues.

If $\lambda = -k^2 < 0$, then $y(x) = a\cosh kx + b\sinh kx$. The boundary conditions force the two equations

$$a + 2bk = 0, \quad (3\cosh 2k + 2k\sinh 2k)a + (3\sinh 2k + 2k\cosh 2k) = 0.$$

This is a homogeneous linear system for a and b and it will have a nonzero solution when the determinant of the coefficient matrix is zero, i.e.,

$$\tan 2k = \frac{4k}{3 - 4k^2}.$$

This equation has nonzero solutions at $k \approx \pm 0.42$. Therefore there is one negative eigenvalue $\lambda \approx -0.42^2 = -0.176$. (This nonlinear equation for k can be solved graphically using a calculator, or using a computer algebra package, or using the solver routine on a calculator).

Exercise 5. When $\lambda = 0$ the ODE is $y'' = 0$, giving $y(x) = Ax + B$. Now apply the boundary conditions to get

$$B - aA = 0, \ \ Al + B + bA = 0.$$

This homogeneous system has a nonzero solution for A and B if and only if $a + b = -abl$. (Note that the determinant of the coefficient matrix must be zero).

Exercise 7. Multiply the equation by y and integrate from $x = 0$ to $x = l$ to get

$$-\int_0^l yy''dx + \int_0^l qy^2 dx = \lambda \int_0^l y^2 dx.$$

Integrate the first integral by parts; the boundary term will be zero from the boundary conditions; then solve for λ to get

$$\lambda = \frac{\int_0^l (y')^2 dx + \int_0^l qy^2 dx}{||y||^2}.$$

Clearly (note $y(x) \neq 0$) the second integral in the numerator and the integral in the denominator are positive ,and thus $\lambda > 0$. $y(x)$ cannot be constant because the boundary conditions would force that constant to be zero.

Exercise 9. When $\lambda = 0$ the ODE is $-y'' = 0$ which gives $y(x) = ax + b$. The boundary conditions force $a = 0$ but do not determine b. Thus $\lambda = 0$ is an eigenvalue with corresponding constant eigenfunctions. When $\lambda = -k^2 < 0$ then the ODE has general solution $y(x) = ae^{kx} + be^{-kx}$, which are exponentials. Easily, exponential functions cannot satisfy periodic boundary conditions, so there are no negative eigenvalues. If $\lambda = k^2 > 0$ then $y(x) = a \sin kx + b \cos kx$. Then $y'(x) = ak \cos kx - bk \sin kx$. Applying the boundary conditions

$$b = a \sin kl + b \cos kl, \quad a = a \cos kl - b \sin kl.$$

We can rewrite this system as a homogeneous system

$$\begin{aligned} a \sin kl + b(\cos kl - 1) &= 0 \\ a(\cos kl - 1) - b \sin kl &= 0. \end{aligned}$$

A homogeneous system will have a nontrivial solution when the determinant of the coefficient matrix is zero, which is in this case reduces to the equation

$$\cos kl = 0.$$

Therefore kl must be a multiple of 2π, or

$$\lambda_n = (2n\pi/l)^2, \quad n = 1, 2, 3, \ldots.$$

The corresponding eigenfunctions are

$$y_n(x) = a_n \sin(2n\pi x/l) + b_n \cos(2n\pi x/l).$$

Exercise 10. Letting $c(x,t) = y(x)g(t)$ leads to the periodic boundary value problem

$$-y'' = \lambda y, \quad y(0) = y(2l),\ y'(0) = y'(2l)$$

and the differential equation

$$g' = \lambda D g,$$

which has solution

$$g(t) = e^{-D\lambda t}.$$

The eigenvalues and eigenfunctions are found exactly as in the solution of Exercise 4, Section 3.4, with l replaced by $2l$. They are

$$\lambda_0 = 0, \lambda_n = n^2\pi^2/l^2, \ \ n = 1, 2, \ldots$$

and

$$y_0(x) = 1, \quad y_n(x) = a_n \cos(n\pi x/l) + b_n \sin(n\pi x/l), \quad n = 1, 2, \ldots.$$

Thus we form

$$c(x,t) = a_0/2 + \sum_{n=1}^{\infty} e^{-n^2\pi^2 Dt/l^2}(a_n \cos(n\pi x/l) + b_n \sin(n\pi x/l).$$

Now the initial condition gives

$$c(x,0) = f(x) = a_0/2 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x/l) + b_n \sin(n\pi x/l),$$

which is the Fourier series for f. Thus the coefficients are given by

$$a_n = \frac{1}{l}\int_0^2 lf(x)\cos(n\pi x/l)dx, \quad n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{1}{l}\int_0^2 lf(x)\sin(n\pi x/l)dx, \quad n = 1, 2, \ldots.$$

Exercise 11. The operator on the left side of the equation has variable coefficients and the ODE cannot be solved analytically in terms of simple functions.

Exercise 12. This problem models the transverse vibrations of a string of length l when the left end is fixed (attached) and the right end experience no force; however, the right end can move vertically. Initially the string is displaced by $f(x)$ and it is not given an initial velocity.

Substituting $u = y(x)g(t)$ into the PDE and boundary conditions gives the SLP

$$-y'' = \lambda y, \quad y(0) = y'(l) = 0$$

and, for g, the equation

$$g'' + c^2\lambda g = 0.$$

The SLP has eigenvalues and eigenfunctions

$$\lambda_n = ((2n+1)\pi/l)^2, \quad y_n(x) = \sin((2n+1)\pi x/l), \quad n = 0, 1, 2, \ldots,$$

and the equation for g has general solution

$$g_n(t) = a_n \sin((2n+1)\pi ct/l) + b_n \cos((2n+1)\pi ct/l).$$

Then we form

$$u(x,t) = \sum_{n=0}^{\infty}(a_n \sin((2n+1)\pi ct/l) + b_n \cos((2n+1)\pi ct/l))\sin((2n+1)\pi x/l).$$

Applying the initial conditions,

$$u(x,t) = f(x) = \sum_{n=0}^{\infty} b_n \sin((2n+1)\pi x/l),$$

which yields

$$b_n = \frac{1}{||\sin((2n+1)\pi x/l)||^2} \int_0^l f(x)\sin((2n+1)\pi x/l).$$

And

$$u_t(x,0) = 0 = \sum_{n=0}^{\infty} a_n c\lambda_n \sin((2n+1)\pi x/l),$$

which gives $a_n = 0$. Therefore the solution is

$$u(x,t) = \sum_{n=0}^{\infty} b_n \cos((2n+1)\pi ct/l)) \sin((2n+1)\pi x/l).$$

Exercise 13. The flux at $x = 0$ is $\phi(0,t) \equiv -u_x(0,t) = -a_0 u(0,t) > 0$, so there is heat flow into the bar and therefore adsorption. At $x = 1$ we have $\phi(1,t) = -u_x(1,t) = a_1 u(1,t) > 0$, and therefore heat is flowing out of the bar, which is radiation. The right side of the inequality $a_0 + a_1 > -a_0 a_1$ is positive, so the positive constant a_1, which measures radiation, must greatly exceed the negative constant a_0, which measures adsorption.

In this problem substituting $u = y(x)g(t)$ leads to the Sturm-Liouville problem

$$-y'' = \lambda y, \quad y'(0) - a_0 y(0) = 0,\ y'(1) + a_1 y(1) = 0$$

and the differential equation

$$g' = \lambda g.$$

There are no nonpositive eigenvalues. If we take $\lambda = k^2 > 0$ then the solutions are

$$y(x) = a\cos kx + b\sin kx.$$

Applying the two boundary conditions leads to the nonlinear equation

$$\tan k = \frac{(a_0 + a_1)k}{k^2 - a_0 a_1}.$$

To determine the roots k, and thus the eigenvalues $\lambda = k^2$, we can graph both sides of this equation to observe that there are infinitely many intersections occurring at k_n, and thus there are infinitely many eigenvalues $\lambda_n = k_n^2$. The eigenfunctions are

$$y_n(x) = \cos k_n x + \frac{a_0}{k_n}\sin k_n x.$$

So the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 t}(\cos k_n x + \frac{a_0}{k_n}\sin k_n x).$$

The c_n are then the Fourier coefficients

$$c_n = (f, y_n)/||y_n||^2.$$

If $a_0 = -1/4$ and $a_1 = 4$ then

$$\tan k = \frac{3.75k}{k^2+1}.$$

From a graphing calculator, the first four roots are approximately $k_1 = 1.08,\ k_2 = 3.85,\ k_3 = 6.81,\ k_4 = 9.82$.

4.3 Generalization and Singular Problems

Exercise 1. (a) Yes (divide by x^2). (b) No.

Exercise 2. Multiply by y and then integrate. Then integrate the first term by parts and apply the boundary conditions.

Exercise 3. Rewrite the equation as

$$x^2y'' + xy' + \lambda y = 0,$$

which is a Cauchy-Euler equation. The roots of the characteristic equation are $m = \pm\sqrt{\lambda}$, where $\lambda = k^2 > 0$. So the general solution is $y(x) = A\cos(k\ln x) + B\sin(k\ln x)$. (Note that a simple energy argument shows the eigenvalues are positive.) Now, $y(1) = 0$ forces $A = 0$, and $y(b) = B\sin(k\ln b) = 0$. Therefore, $k\ln b = n\pi,\ n = 1,2,3\ldots$. Thus the eigenvalues are $\lambda_n = n\pi/\ln b$ and the eigenfunctions are $y_n(x) = \sin n\pi,\ n = 1,2,3,\ldots$.

Exercise 4. Multiplying the differential equation by y and integrating from $x = 1$ to $x = \pi$ gives

$$-\int_1^{\pi} y(x^2y')'dx = \lambda\int_1^{\pi} y^2dx$$

or, upon integrating the left side by parts,

$$-x^y y' \mid_1^{\pi} - \int_1^{\pi} x^2(y')^2dx = \lambda\int_1^{\pi} y^2dx.$$

The boundary term vanishes because of the boundary conditions. Therefore, because both integrals are nonnegative we have $\lambda \geq 0$. If $\lambda = 0$ then $y' =$const$=$ 0 (by the boundary conditions). So $\lambda \neq 0$ and the eigenvalues are therefore positive.

If $\lambda = k^2 > 0$, then the ODE becomes

$$x^2 y'' + 2xy' + k^2 y = 0,$$

which is a Cauchy-Euler equation (see the Appendix on differential equations in the text). This can be solved to determine eigenvalues

$$\lambda_n = \left(\frac{n\pi}{\ln \pi}\right)^2 + \frac{1}{4}$$

with corresponding eigenfunctions

$$y_n(x) = \frac{1}{\sqrt{x}} \sin\left(\frac{n\pi}{\ln \pi} \ln x\right).$$

Exercise 7. Letting $u = g(t)y(x)$ and substituting into the equation and boundary conditions gives $g'' = \lambda g$ and the Sturm–Liouville problem

$$-y'' = -\lambda \frac{1}{c(x)^2} y, \quad y(0) = y(l) = 0.$$

The weight function is $1/c(x)^2$.

Exercise 8. We have $\rho(x)u_{tt} = u_{xx}$. Putting $u = Y(x)g(t)$ gives, upon separating variables,

$$-y'' = \rho(x)\lambda y, \quad y(0) = y(1) = 0.$$

We have

$$-y_f'' = \rho(x)\lambda_f y_f, \quad y_f(0) = y_f(1) = 0.$$

Integrating from $x = 0$ to $x = s$ gives

$$-y_f'(s) + y_f'(0) = \lambda_f \int_0^s \rho(x) y_f(x) dx.$$

Now integrate from $s = 0$ to $s = 1$ to get

$$\begin{aligned} y_f'(0) &= \lambda_f \int_0^1 \int_0^s \rho(x) y_f(x) dx ds \\ &= \lambda_f \int_0^1 (1 - x)\rho(x) y_f(x) dx. \end{aligned}$$

The last step follows by interchanging the order of integration. If $\rho(x) =_r ho_0$ is a constant, then

$$\rho_0 = \frac{y_f'(0)}{\lambda_f \int_0^1 (1 - x) y_f(x) dx}.$$

Exercise 9. From Exercise 2 in Section 4.6 we have the solution

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{kn^2}(1 - e^{-n^2kt}) \left(\int_0^{\pi} f(r) \sin nr \, dr \right) \sin nx.$$

Therefore

$$U(t) = u(\pi/2, t) = \int_0^{\pi} \left(\frac{2}{\pi} \sum_{n=1}^{\infty} \sin nr \sin(n\pi/2) \frac{1}{kn^2}(1 - e^{-n^2kt}) \right) f(r) dr.$$

We want to recover $f(x)$ if we know $U(t)$. This problem is not stable, as the following example shows. Let

$$u(x,t) = m^{-3/2}(1 - e^{-m^2t}) \sin mx, \quad f(x) = \sqrt{m} \sin mx.$$

This pair satisfies the model. If m is sufficiently large, then$u(x,t)$ is uniformly small; yet $f(0)$ is large. So a small error in measuring $U(t)$ will result in a large change in $f(x)$.

4.4 Laplace's Equation

Exercise 1. Substituting $u(x,y) = \phi(x)\psi(y)$ we obtain the Sturm-Liouville problem

$$-\phi'' = \lambda\phi, \ \ x \in (0,l); \ \ \phi(0) = \phi(l) = 0$$

and the differential equation

$$\psi'' - \lambda\psi = 0.$$

The SLP has eigenvalues and eigenfunctions

$$\lambda_n = n^2\pi^2/l^2, \quad \phi_n(x) = \sin(n\pi x/l),$$

and the solution to the ψ–equation is

$$\psi_n(y) = a_n \cosh(n\pi y/l) + b_n \sinh(n\pi y/l).$$

Therefore

$$u(x,y) = \sum_{n=1}^{\infty} (a_n \cosh(n\pi y/l) + b_n \sinh(n\pi y/l)) \sin(n\pi x/l).$$

Now we apply the boundary conditions:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/l) = 0,$$

so $a_n = 0$. Therefore

$$u(x,1) = G(x) = \sum_{n=1}^{\infty} b_n \sinh(n\pi/l)) \sin(n\pi x/l).$$

Thus

$$b_n \sinh(n\pi/l) = \frac{2}{l}\int_0^{\pi} G(x)\sin(n\pi x/l)dx,$$

which determines the coefficients b_n.

Exercise 2. This problem models the steady state temperatures in a rectangular plate that is insulated on both sides, whose temperature is zero on the top, and whose temperature is $f(x)$ along the bottom. Letting $u = g(y)\phi(x)$ and substituting into the PDE and boundary conditions gives the Sturm-Liouville problem

$$-\phi'' = \lambda\phi, \quad \phi'(0) = \phi'(a) = 0$$

and the differential equation

$$g'' - \lambda g = 0.$$

The eigenvalues and eigenfunctions are $\lambda_0 = 0,\ \phi(x) = 1$ and

$$\lambda_n = n^2\pi^2/a^2, \quad \phi_n(x) = \cos(n\pi x/a), \quad n = 1, 2, 3, \ldots.$$

The solution to the g equation is, corresponding to the zero eigenvalue, $g_0(y) = c_0 y + d_0$, and corresponding to the positive eigenvalues,

$$g_n(y) = c_n \sinh(n\pi y/a) + d_n \cosh(n\pi y/a).$$

Thus we form the linear combination

$$u(x,y) = c_0 y + d_0 + \sum_{n=1}^{\infty}(c_n \sinh(n\pi y/a) + d_n \cosh(n\pi y/a))\cos(n\pi x/a).$$

Now apply the boundary conditions on y to compute the coefficients:

$$u(x,0) = f(x) = d_0 + \sum_{n=1}^{\infty} d_n \cos(n\pi x/a),$$

which gives

$$d_0 = \frac{1}{a}\int_0^a f(x)dx, \quad d_n = \frac{2}{a}\int_0^a f(x)\cos(n\pi x/a)dx.$$

Next

$$u(x,b) = 0 = c_0 b + d_0 + \sum_{n=1}^{\infty}(c_n \sinh(n\pi b/a) + d_n \cosh(n\pi b/a))\cos(n\pi x/a).$$

Therefore

$$c_0 = -d_0/b, \quad c_n = -\frac{\cosh(n\pi b/a)}{\sinh(n\pi b/a)} d_n.$$

Exercise 3. The general solution is given in the text. Here

$$f(\theta) = 4 + 3\sin\theta.$$

The right side is its Fourier series, so the Fourier coefficients are given by

$$\frac{a_0}{2} = 4, \quad Rb_1 = 3,$$

with all the other Fourier coefficients identically zero. So the solution is

$$u(r,\theta) = 4 + \frac{3r}{R}\sin\theta.$$

Exercise 5. The exterior Poisson formula applies but the integral is difficult. In this simple case assume a separable solution of the form $u(r,\theta) = Ar^{-n}\cos\theta$. Substitute into the radial form of the equation to get $n = 1$ (to get bounded solutions). Apply the boundary condition to get $A = 1$. Therefore

$$u(r,\theta) = \frac{1}{r}\cos\theta.$$

Exercise 6. Substituting $u(r,\theta) = g(\theta)y(r)$ into the PDE and boundary conditions gives the Sturm-Liouville problem

$$-g'' = \lambda g, \quad g(0) = g(\pi/2) = 0$$

and the differential equation

$$r^2y'' + ry' + \lambda y = 0.$$

This SLP has been solved many times in the text and in the problems. The eigenvalues and eigenfunctions are

$$\lambda_n = 4n^2, \quad g_n(\theta) = \sin(2n\theta), \quad n = 1, 2, \ldots.$$

The y equation is a Cauchy-Euler equation and has bounded solution

$$y_n(r) = r^{2n}.$$

Form

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n r^{2n}\sin(2n\theta).$$

Then the boundary condition at $r = R$ gives

$$u(R, \theta) = f(\theta) = \sum_{n=1}^{\infty} b_n R^{2n} \sin(2n\theta).$$

Hence the coefficients are

$$b_n = \frac{1}{\pi R^{2n}} \int_0^{\pi/2} f(\theta) \sin(2n\theta) d\theta.$$

Exercise 7. Substituting $u(r, \theta) = g(\theta)y(r)$ into the PDE and boundary conditions gives the Sturm-Liouville problem

$$-g'' = \lambda g, \quad g(0) = g'(\pi/2) = 0$$

and the differential equation

$$r^2 y'' + ry' + \lambda y = 0.$$

The eigenvalues and eigenfunctions are

$$\lambda_n = (2n+1)^2, \quad g_n(\theta) = \sin((2n+1)\theta), \quad n = 0, 1, 2, \ldots.$$

The y equation is a Cauchy-Euler equation and has bounded solution

$$y_n(r) = r^{2n+1}$$

Form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^{2n+1} \sin((2n+1)\theta).$$

Then the boundary condition at $r = R$ gives

$$u(R, \theta) = f(\theta) = \sum_{n=1}^{\infty} b_n R^{2n+1} \sin((2n+1)\theta).$$

Hence the coefficients are

$$b_n = \frac{1}{\pi R^{2n+1}} \int_0^{\pi/2} f(\theta) \sin((2n+1)\theta) d\theta.$$

Exercise 10. Let $w = u + v$ where u satisfies the Neumann problem and v satisfies the boundary condition $n \cdot \nabla v = 0$. Then

$$\begin{aligned}
E(w) &= E(u+v)\\
&= \frac{1}{2}\int_\Omega (\nabla u \cdot \nabla u + 2\nabla u \cdot \nabla v + \nabla v \cdot \nabla v)dV - \int_{\partial\Omega}(hu - hv)dA\\
&= E(u) + \int_\Omega \nabla u \cdot \nabla v\, dV + \frac{1}{2}\int_\Omega \nabla v \cdot \nabla v dV - \int_{\partial\Omega} hv\, dA\\
&= E(u) + \int_{\partial\Omega} v\nabla u \cdot n\, dA - \int_\Omega v\Delta u\, dV + \frac{1}{2}\int_\Omega \nabla v \cdot \nabla v dV - \int_{\partial\Omega} hv\, dA\\
&= E(u) + \int_{\partial\Omega} vh\, dA + \frac{1}{2}\int_\Omega \nabla v \cdot \nabla v dV - \int_{\partial\Omega} hv\, dA\\
&= E(u) + \frac{1}{2}\int_\Omega \nabla v \cdot \nabla v dV
\end{aligned}$$

So $E(u) \le E(w)$.

Exercise 11. Integrate the partial differential equation over a region Ω and then use the divergence theorem to get

$$\int_\Omega f dV = \int_\Omega \Delta u dV = \int_{\partial\Omega} \operatorname{grad} u \cdot \mathbf{n} dA = \int_{\partial\Omega} h dA.$$

In a steady heat flow context, for example, this states that net rate that heat is produced by sources in the region must equal the rate that heat leaves through the boundary.

Exercise 12. Take $u = \rho^{-1}w$. (NOTE: there is a typographical error in the exercise.) Then find u_ρ and $u_{\rho\rho}$ and substitute into the PDE to reduce the equation to $w_{\rho\rho} - k^2 w = 0$. This has general solution $w = c_1 e^{k\rho} + c_2 e^{-k\rho}$.

Exercise 13. Multiply both sides of the PDE by u and integrate over Ω. We obtain

$$\int_\Omega u\Delta u\, dV = c\int_\Omega u^2 dV.$$

Now use Green's first identity to obtain

$$\int_{\partial\Omega} u\nabla u \cdot n dA - \int_\Omega \nabla u \cdot \nabla u\, dV = c\int_\Omega u^2 dV,$$

or

$$-\int_{\partial\Omega} au^2 dA - \int_\Omega \nabla u \cdot \nabla u\, dV = c\int_\Omega u^2 dV.$$

The left side is negative and the right side is positive. Then both must be zero, or

$$\int_\Omega u^2 dV = 0.$$

Hence $u = 0$ in Ω.

The uniqueness argument is standard. Let u and v be two solutions to the boundary value problem

$$\Delta u - cu = f, \quad x \in \Omega \qquad n \cdot \nabla u + au = g, \quad x \in \partial\Omega.$$

Then the difference $w = u - v$ satisfies the homogeneous problem

$$\Delta w - cw = 0, \quad x \in \Omega \qquad n \cdot \nabla w + aw = 0, \quad x \in \partial\Omega.$$

By the first part of the problem we know $w = 0$ and therefore $u = v$.

Exercise 14. Let w and v be two solutions and take $u = w - v$. Then $\Delta u - cu = 0$ on Ω and $\mathbf{n} \cdot \operatorname{grad} u + au = 0$ on $\partial\Omega$. Now use Exercise 13 to get $u = 0$, so that $w = v$.

Exercise 15. Multiplying the equation $\Delta u = 0$ by u, integrating over Ω, and then using Green's identity gives

$$\int_\Omega u\Delta u \, dV = \int_{\partial\Omega} u\nabla u \cdot n dA - \int_\Omega \nabla u \cdot \nabla u dV = 0.$$

Thus

$$\int_\Omega \nabla u \cdot \nabla u dV = 0,$$

which implies

$$\nabla u = 0.$$

Thus $u = \text{constant}$.

Exercise 16. Let v and w be two solutions and take $u = v - w$. To show $u = 0$ we use an energy argument. Multiply the PDE by u and integrate to get $\int_\Omega u\Delta u \, dV = 0$. By Green's first identity,

$$\int_{\partial\Omega} u\nabla u \cdot \mathrm{n} dA - \int_\Omega \nabla u \cdot \nabla u dV = 0.$$

Using the boundary condition, we get

$$-\int_{\partial\Omega} au^2 dA - \int_\Omega \nabla u \cdot \nabla u dV = 0.$$

Both terms on the left are negative, which is a contradiction. So $u = 0$ on Ω.

Exercise 18. *Typographical error*: the boundary condition should be $du/dn = \sin^2\theta$. We have

$$\int_\Omega f dV = 0 \quad \text{and} \quad \int_{\partial\Omega} \sin^2\theta dA > 0,$$

which contradicts the result of Exercise 11.

Exercise 19. The radial equation in spherical coordinates is $\rho^{-2}(\rho^2 u_\rho)_\rho = 0$ has solution $u(\rho) = c_1\rho^{-1} + c_2$. Apply the boundary conditions to determine c_1 and c_2.

4.5 Cooling of a Sphere

Exercise 1. The problem is

$$-y'' - \frac{2}{\rho}y' = \lambda y, \quad y(0) \text{ bounded}, \ y(\pi) = 0.$$

Making the transformation $Y = \rho y$ we get $-Y'' = \lambda Y$. If $\lambda = -k^2 < 0$ then

$$Y = a\sinh k\rho + b\cosh k\rho,$$

or

$$y = \rho^{-1}(a\sinh k\rho + b\cosh k\rho).$$

For boundedness at $\rho = 0$ we set $b = 0$. Then $y(\pi) = 0)$ forces $\sinh k\pi = 0$. Thus $k = 0$. Consequently, there are no negative eigenvalues.

Exercise 2. From the formula developed in the text the temperature at $\rho = 0$ is

$$u(0,t) = 74\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}e^{-n^2kt},$$

where $k = 5.58$ inches-squared per hour.

Exercise 3. The boundary value problem is

$$u_t = k(u_{\rho\rho} + \frac{2}{\rho}u_\rho),$$
$$u_\rho(R,t) = -hu(R,t), \quad t > 0,$$
$$u(\rho,0) = f(\rho), \quad 0 \le \rho \le R.$$

Assume $u = y(\rho)g(t)$. Then the PDE and boundary conditions separate into the boundary value problem

$$y'' + (2/\rho)y' + \lambda y = 0, \quad y'(R) = -hy(R), \ \ y \text{ bounded}$$

and the differential equation

$$g- = -\lambda kg.$$

The latter has solution $g(t) = exp(-\lambda kt)$. One can show that the eigenvalues are positive. So let $\lambda = p^2$ and make the substitution $Y = \rho y$, as in the text, to obtain

$$Y'' + p^2Y = 0.$$

This has solution

$$Y(\rho) = a\cos p\rho + b\sin p\rho.$$

But $Y(0) = 0$ forces $a = 0$ (because y is bounded). Then the other boundary condition forces p to satisfy the nonlinear equation

$$\tan Rp = \frac{Rp}{1 - Rh}.$$

If we graph both sides of this equation against p we note that there are infinitely many intersections, giving infinitely many roots $p_n,\ n = 1, 2, \ldots$, and therefore infinitely many eigenvalues $\lambda_n = p_n^2$. The corresponding eigenfunctions are

$$y_n(\rho) = \rho^{-1}\sin(\sqrt{\lambda_n}\rho).$$

Thus we have

$$u(\rho, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n kt}\rho^{-1}\sin(\sqrt{\lambda_n}\rho).$$

The c_n are found from the initial condition. We have

$$u(\rho, 0) = f(\rho) = \sum_{n=1}^{\infty} c_n \rho^{-1}\sin(\sqrt{\lambda_n}\rho).$$

Thus

$$c_n = \frac{\int_0^R \rho f(\rho)\sin(\sqrt{\lambda_n}\rho)d\rho}{\int_0^R \sin^2(\sqrt{\lambda_n}\rho)\rho d\rho}.$$

Exercise 4. Representing the Laplacian in spherical coordinates, the boundary value problem for $u = u(\rho, \phi)$, where $\rho \in (0, 1)$ and $\phi \in (0, \pi)$, is

$$\Delta u = u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{\rho^2 \sin\phi}(\sin\phi u_\phi) = 0,$$
$$u(1, \phi) = f(\phi), \quad 0 \le \phi \le \pi.$$

Observe, by symmetry of the boundary condition, u cannot depend on the angle θ. Now assume $u = R(\rho)Y(\phi)$. The PDE separates into two equations,

$$\rho^2 R'' + 2\rho R' - \lambda R = 0$$

and

$$\frac{1}{\sin\phi}(\sin\phi Y')' = \lambda Y.$$

We transform the Y equation by changing the independent variable to $x = \cos\phi$. Then we get, using the chain rule,

$$\frac{1}{\sin\phi}\frac{d}{d\phi} = -\frac{d}{dx}.$$

So the Y–equation becomes

$$-\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) = \lambda y \quad -1 < x < 1.$$

By the given facts, this equation has bounded, orthogonal, solutions $y_n(x) = P_n(x)$ on $[-1, 1]$ when $\lambda = \lambda_n = n(n+1)$, $n = 0, 1, 2, \ldots$. Here $P_n(x)$ are the Legendre polynomials.

Now, the R–equation then becomes

$$\rho^2 R'' + 2\rho R' - n(n+1)R = 0$$

This is a Cauchy-Euler equation (see the Appendix in the text) with characteristic equation

$$m(m-1) + m - n(n+1) = 0.$$

The roots are $m = n, -(n+1)$. Thus

$$R_n(\rho) = a_n \rho^n$$

are the bounded solutions (the other root gives the solution ρ^{-n-1}, which is unbounded at zero). Therefore we form

$$u(\rho, \phi) = \sum_{n=0}^{\infty} a_n \rho^n P_n(\cos\phi)$$

or, equivalently,

$$u(\rho, x) = \sum_{n=0}^{\infty} a_n \rho^n P_n(x).$$

Applying the boundary condition gives the coefficients. We have

$$u(1, x) = f(\arccos x) = \sum_{n=0}^{\infty} a_n P_n(x).$$

By orthogonality we get

$$a_n = \frac{1}{||P_n||^2}\int_{-1}^{1} f(\arccos x) P_n(x) dx,$$

or

$$a_n = \frac{1}{||P_n||^2}\int_{0}^{\pi} f(\phi) P_n(\cos\phi)\sin\phi \, d\phi.$$

By direct differentiation we get

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1),\ P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x.$$

Also the norms are given by

$$||P_0||^2 = 2, \ \ ||P_1||^2 = \frac{2}{3}, \ \ ||P_2||^2 = \frac{2}{5}.$$

When $f(\phi) = \sin\phi$ the first few Fourier coefficients are given by

$$a_0 = \frac{\pi}{4}, \ \ a_1 = a_3 = 0, \ \ a_2 = -0.49.$$

Therefore a two-term approximation is given by

$$u(\rho, \phi) \approx \frac{\pi}{4} - \frac{0.49}{2}\rho^2(3\cos^2\phi - 1).$$

Exercise 5. To determine the temperature of the earth we must derive the temperature formula for any radius R (the calculation in the text uses $R = \pi$). The method is exactly the same, but now the eigenvalues are $\lambda_n = n^2\pi^2/R^2$ and the eigenfunctions are $y_n = \rho^{-1}\sin(n\pi\rho/R)$, for $n = 1, 2, \ldots$. Then the temperature is

$$u(\rho, t) = \frac{2RT_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-lam_n kt}\rho^{-1}\sin(n\pi\rho/R).$$

Now we compute the geothermal gradient at the surface, which is $u_\rho(\rho, t)$ at $\rho = R$. We obtain

$$u_\rho(R, t) = -\frac{2T_0}{R}\sum_{n=1}^{\infty}e^{-n^2\pi^2kt/R^2}.$$

If G is the value of the geothermal gradient at the current time $t = t_c$, then

$$\frac{RG}{2T_0} = \sum_{n=1}^{\infty}e^{-n^2\pi^2kt_c/R^2}.$$

We must solve for t_c. Notice that the sum has the form

$$\sum_{n=1}^{\infty}e^{-an^2}$$

where $a = \pi^2kt_c/R^2$. We can make an approximation by noting that the sum represents a Riemann sum approximation to the integral

$$\int_0^{\infty}e^{-ax^2}dx = \frac{1}{2}\sqrt{\pi/a}.$$

So we use this value to approximate the sum, i.e.,

$$\sum_{n=1}^{\infty}e^{-n^2\pi^2kt_c/R^2} \approx \frac{R}{2\sqrt{k\pi t_c}}.$$

Solving for t_c gives

$$t_c = \frac{T_0^2}{G^2 k\pi}.$$

Substituting the numbers in from Exercise 5 in Section 2.4 gives $t_c = 5.15(10)^8$ years. This is the same approximation we found earlier.

4.6 Diffusion in a Disk

Exercise 1. The differential equation is $-(ry')' = \lambda ry$. Multiply both sides by y and integrate over $[0, R]$ to get

$$\int_0^R -(ry')' y dr = \lambda \int_0^R ry^2 dr.$$

Integrating the left hand side by parts gives

$$-ryy' \mid_0^R \int_0^R r(y')^2 dr = \lambda \int_0^R ry^2 dr.$$

But, since y and y' are assumed to be bounded, the boundary term vanishes. The remaining integrals are nonnegative and so $\lambda \geq 0$.

Exercise 2. Let y, λ and w, μ be two eigenpairs. Then $-(ry')' = \lambda ry$ and $-(rw')' = \mu rw$. Multiply the first of these equations by w and the second by y, and then subtract and integrate to get

$$\int_0^R [-(ry')' w + (rw')' y] dr = (\lambda - \mu) \int_0^R ruw dr.$$

Now integrate both terms in the first integral on the left hand side by parts to get

$$(-ry'w + rw'y) \mid_0^R + \int_0^R (ry'w' - rw'y') dr = (\lambda - \mu) \int_0^R ruw dr.$$

The left side of the equation is zero and so y and w are orthogonal with respect to the weight function r.

Exercise 3. We have

$$u(r,t) = \sum_{n=1}^{\infty} c_n e^{-0.25\lambda_n t} J_0(z_n r),$$

where

$$c_n = \frac{\int_0^1 5r^4(1-r)J_0(z_n r)dr}{\int_0^1 J_0(z_n r)^2 r dr}.$$

We have $z_1 = 2.405,\ z_2 = 5.520,\ z_3 = 8.654$. Use a computer algebra program to calculate a 3-term approximation.

4.7 Sources on Bounded Domains

Exercise 1. Use Duhamel's principle to solve the problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x,t), \quad 0 < x < \pi,\ t > 0, \\ u(0,t) = u(\pi,t) &= 0, \quad t > 0, \\ u(x,0) = u_t(x,0) &= 0, \quad 0 < x < \pi. \end{aligned}$$

Consider the problem for $w = w(x,t,\tau)$, where τ is a parameter:

$$\begin{aligned} w_{tt} - c^2 w_{xx} &= 0, \quad 0 < x < \pi,\ t > 0, \\ w(0,t,\tau) = w(\pi,t,\tau) &= 0, \quad t > 0, \\ w(x,0,\tau) = 0,\ w_t(x,0,\tau) &= 0, \quad 0 < x < \pi. \end{aligned}$$

This problem was solved in Section 4.1 (see (4.14)–(4.14)). The solution is

$$w(x,t,\tau) = \sum_{n=1}^{\infty} c_n(\tau) \sin nct \sin nx,$$

where

$$c_n(\tau) = \frac{2}{nc\pi} \int_0^{\pi} f(x,\tau) \sin nx\ dx.$$

So the solution to the original problem is

$$u(x,t) = \int_0^t w(x, t-\tau, \tau) d\tau.$$

Exercise 3. If $f = f(x)$, and does not depend on t, then the solution can be written

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\int_0^{\pi} f(r) \sin nr\ dr \right) (\left(\int_0^t e^{-n^2 k(t-\tau)} d\tau \right) \sin nx.$$

But a straightforward integration gives

$$\int_0^t e^{-n^2 k(t-\tau)} d\tau = \frac{1}{kn^2}(1 - e^{-n^2 kt}).$$

Therefore

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{kn^2}(1-e^{-n^2kt})\left(\int_0^{\pi} f(r)\sin nr\ dr\right)\sin nx.$$

Taking the limit as $t \to \infty$ gives

$$U(x) \equiv \lim u(x,t) = u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{kn^2}\left(\int_0^{\pi} f(r)\sin nr\ dr\right)\sin nx.$$

Now consider the steady state problem

$$-kv'' = x(\pi - x), \quad v(0) = v(\pi) = 0.$$

This can be solved directly by integrating twice and using the boundary conditions to determine the constants of integration. One obtains

$$v(x) = -\frac{1}{12k}(2\pi x^2 - x^4 - \pi^3 x).$$

To observe that the solution $v(x)$ is the same as the limiting solution $U(x)$ we expand the right side of the v–equation in its Fourier sine series on $[0,\pi]$. Then

$$-kv'' = \sum_{n=1}^{\infty} c_n \sin nx,$$

where

$$c_n = \frac{2}{\pi}\int_0^{\pi} r(\pi - r)\sin nr dr.$$

Integrating the differential equation twice gives

$$-kv(x) + kv'(0)x = \sum_{n=1}^{\infty} c_n n^{-2}(\sin nx - x).$$

Evaluating at $x = \pi$ gives

$$kv'(0) = \sum_{n=1}^{\infty} c_n n^{-2}.$$

Whence

$$kv(x) = \sum_{n=1}^{\infty} c_n n^{-2}\sin nx,$$

or

$$v(x) = \frac{2}{k\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}\left(\int_0^{\pi} f(r)\sin nr\ dr\right)\sin nx.$$

Hence $v(x) = U(x)$.

Exercise 4. Homogenize the boundary conditions by letting $w(x,t) = u(x,t) - Ax$. Then the equation for w is $w_t = w_{xx}$ with $w(0,t) = w(1,t) = 0$ and $w(x,0) = \cos x - Ax$. The w-problem is a straightforward separation of variables problem with homogeneous equation and boundary conditions.

Exercise 5. Refer to Remark 4.29. First find the solution of the steady-state problem which is $U''(x) = -Q$, $U(0) = 0$, $U(1) = 2u_0$. This is easily solved to get

$$U(x) = -\frac{1}{2}x^2 + (2u_0 + \frac{1}{2}Q)x.$$

Now take $w(x,t) = u(x,t) - U(x)$. It is easily seen that w satisfies the problem $w_t = w_{xx}$ with $w(0,t) = w(1,t) = 0$ and $w(x,0) = u_0(1 - \cos \pi x) - U(x)$, which is a standard separation of variables problem.

Exercise 6. Follow the steps in Remark 4.30.

Exercise 7. Again, use Remark 4.29; the nonhomogeneous equation has a source term that depends only on x. The steady-state solution satisfies $U''(x) = -\frac{1}{k}\sin 3\pi x$, with $U(0) = U(1) = 0$. By integrating twice,

$$U(x) = \frac{1}{9\pi^2 k}\sin 3\pi x.$$

Take $w(x,t) = u(x,t) - U(x)$. Then w satisfies a homogenous problem $w_t = w_{xx}$ with $w(0,t) = w(1,t) = 0$, and the initial condition is $w(x,0) = \sin \pi x - U(x)$. This is solved by separating variables.

Exercise 8. *Typographical error*: The boundary condition at $R = R_1$ is $u = u_1$. Follow Example 4.31.

Exercise 9. Follow Example 4.32.

Exercise 10. The problem is

$$\begin{aligned} u_t &= \Delta u + f(r,t) \quad 0 \le r < R,\ t > 0, \\ u(R,t) &= 0, \quad t > 0, \\ u(r,0) &= 0, \quad 0 < r < R. \end{aligned}$$

For $w = w(r,t,\tau)$ we consider the problem

$$\begin{aligned} w_t &= \Delta w \quad 0 \le r < R,\ t > 0, \\ w(R,t,\tau) &= 0, \quad t > 0, \\ w(r,0,\tau) &= f(r,\tau), \quad 0 < r < R. \end{aligned}$$

This is the model for heat flow in a disk of radius R; the solution is given by equation (4.53) in Section 4.5 of the text. It is

$$w(r,t,\tau) = \sum c_n(\tau)e^{-\lambda_n kt}J_0(z_n r/R),$$

where z_n are the zeros of the Bessel function J_0, $\lambda_n = z_n^2/R^2$ and

$$c_n(\tau) = \frac{1}{||J_0(z_n r/R)||^2}\int_0^R f(r,\tau)J_0(z_n r/R)r dr.$$

Then

$$u(r,t) = \int_0^R w(r,t-\tau,\tau)d\tau.$$

Exercise 11. The source term depends only on r, so take $w(r,t) = u(r,t) - U(r)$, where $U(r)$ is the steady-state solution to $r^{-1}(rU')' = -f(r)$, $U(R) = u_0$. We obtain

$$U(r) = \int_r^R \frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma\, d\rho + u_0.$$

Then w satisfies a homogeneous differential equation and boundary condition.

4.8 Poisson's Equation

Exercise 1. $u(r,\theta) = (1/\sqrt{2})r\sin\theta$.

Exercise 2. There is no θ dependence in the problem, so take $u = u(r)$. Substituting into the equation gives $(ru')' = -Ar$. Integrating twice gives $u(r) = -\frac{1}{4}Ar^2 + c_1\ln r + c_2$. For boundedness, take $c_1 = 0$. Applying the boundary condition at R gives $c_2 = 1 + AR^2/4$. Thus

$$u = \frac{1}{4}(R^2 - r^2) + 1.$$

Exercise 4. The solution depends only on r. Try $u = U(r)$, where U is to be determined.

Exercise 5. The solution depends only on r. Try $u = U(r)$, where U is to be determined.

Exercise 8. Following the hint in the text we have

$$u(x,y) = \sum_{n=1}^{\infty} g_n(y)\sin nx,$$

where we find

$$g_n'' - n^2 g_n = f_n(y), \quad g_n(0) = g_n(1) = 0.$$

Here, the $f_n(y)$ are the Fourier coefficients of $f(x, y)$. From the variation of parameters formula

$$g_n(y) = ae^{ny} + be^{-ny} - \frac{2}{n}\int_0^y \sinh(n(\xi - y))f_n(\xi)d\xi.$$

Now $g_n(0) = 0$ implies $b = -a$. So we can write

$$g_n(y) = 2a\sinh ny - \frac{2}{n}\int_0^y \sinh(n(\xi - y))f_n(\xi)d\xi$$

which gives, using $g_n(1) = 0$,

$$a = \frac{1}{n\sinh n}\int_0^1 \sinh(n(\xi - 1))f_n(\xi)d\xi.$$

Thus the $g_n(y)$ are given by

$$g_n(y) = \frac{2\sinh ny}{n\sinh n}\int_0^1 \sinh(n(\xi - 1))f_n(\xi)d\xi - \frac{2}{n}\int_0^y \sinh(n(\xi - y))f_n(\xi)d\xi.$$

5

Applications in The Life Sciences

5.1 Age-Structured Models

Exercise 1. Write

$$1 = \int_3^8 4e^{-(r+0.03)a} da = -\frac{4}{r+0.03} \left(e^{-8(r+0.03)} - e^{-3(r+0.03)} \right),$$

and use a software package or calculator to solve for r.

Exercise 2. First note that $u(a,t) = 0$ for $a > t + \delta$, since $f(a) = 0$ for $a > \delta$. For (a) observe that the renewal equation (5.9) is

$$B(t) = \int_0^t \beta B(t-a) e^{-\gamma a} da + \int_0^\infty \beta f(a-t) e^{-\gamma t} da.$$

The first integral becomes, upon changing variables to $s = t - a$,

$$\int_0^t \beta B(t-a) e^{-\gamma a} da = \int_0^t \beta B(s) e^{-\gamma(t-s)} ds.$$

The second integral is

$$\int_0^\infty \beta f(a-t) e^{-\gamma t} da = \int_t^{t+\delta} \beta u_0 e^{-\gamma t} da = \beta u_0 \delta e^{-\gamma t}.$$

For (b), differentiate (using Leibniz rule)

$$B(t) = \int_0^t \beta B(s) e^{-\gamma(t-s)} ds + \beta u_0 \delta e^{-\gamma t}$$

to get

$$B'(t) = \int_0^t \beta B(s)e^{-\gamma(t-s)}ds(-\gamma) + \beta B(t) - \gamma\beta u_0\delta e^{-\gamma t} = (\beta - \gamma)B(t).$$

For (c) note that the last equation is the differential equation for growth-decay and has solution

$$B(t) = B(0)e^{(\beta-\gamma)t}.$$

Therefore the solution from (5.7)–(5.8) is given by

$$u(a,t) = \begin{cases} 0, & a > t + \delta \\ u_0 e^{-\gamma t}, & t < a < t + \delta \\ B(0)e^{(\beta-\gamma)t}e^{-\beta a}, & 0 < a < t. \end{cases}$$

Finally, for part (d) we have, using part (c),

$$\begin{aligned} N(t) &= \int_0^t u(a,t)da + \int_t^{t+\delta} u(a,t)da \\ &= \int_0^t B(0)e^{(\beta-\gamma)t}e^{-\beta a}da + \int_t^{t+\delta} u_0 e^{-\gamma t}da \\ &= \frac{B(0)}{\beta}e^{(\beta-\gamma)t}(1 - e^{-\beta t}) + \delta u_0 e^{-\gamma t}. \end{aligned}$$

Exercise 3. Integrate the PDE from $a = 0$ to $a = \infty$ to get

$$N(t) = -\int_0^\infty u_a da - m(N)N = B(t) - m(N)N.$$

To get an equation for B we differentiate the $B(t)$ equation to get

$$\begin{aligned} B'(t) &= \int_0^\infty b_0 e^{-\gamma a}u_t da = \int_0^\infty b_0 e^{-\gamma a}(-u_a - m(N)u)da \\ &= -m(N)B(t) - \int_0^\infty b_0 e^{-\gamma a}(u_a)da \\ &= -m(N)B(t) - \left[b_0 e^{-\gamma a}u|_0^\infty + \int_0^\infty b_0\gamma e^{-\gamma a}u da\right] \\ &= -m(N)B(t) - b_0 B(t) - \gamma B(t). \end{aligned}$$

To obtain the next-to-last line we used integration by parts. In summary we have the dynamical system

$$\begin{aligned} N' &= B - m(N)N, \\ B' &= (b_0 - \gamma - m(N))B. \end{aligned}$$

In the phase plane the paths or integral curves are defined by

$$\frac{dB}{dN} = \frac{(b_0 - \gamma - m(N))B}{B - m(N)N}.$$

Observe that $B = (b_0 - \gamma)N$ is easily shown to be a solution to this equation. It represents a straight line in the NB plane. The line B=0 is a horizontal nullcline where the vector field points to the left. Another horizontal nullcline is the vertical line $N = N^*$, where N^* is the root of $m(N) = (b_0 - \gamma)$. The point $P = (N^*, (b_0 - \gamma)N^*)$ is an equilibrium that lies on the straight line solution curve $B = (b_0 - \gamma)N$. The solution cannot oscillate since that it would require it cross the straight line, violating uniqueness. On the straight line solution, the direction is toward the point P.

Exercise 5. Let $\xi = a - t, \tau = t$. In these characteristic coordinates the PDE becomes

$$U_\tau = -\frac{c}{d - \xi - \tau}U.$$

Separating variable and integrating gives

$$U = (d - \xi - \tau)^c \varphi(\xi)$$

or

$$u(a, t) = (d - a)^c \varphi(a - t),$$

which is the general solution. Now, for the region $a > t$ we use the initial condition to determine φ. We have

$$u(a, 0) = (d - a)^c \varphi(a) = f(a),$$

which gives $\varphi(a) = f(a)(d - a)^c$. Hence

$$u(a, t) = (d - a)^c f(a - t)(d - a - t)^{-c}, \quad a > t.$$

For the region $a < t$ we use the boundary condition. to determine φ. Thus,

$$u(0, t) = d^c \varphi(-t) = B(t),$$

or

$$\varphi(t) = B(-t)d^{-c}.$$

Whence

$$u(a, t) = (d - a)^c B(t - a)d^{-c}, \quad 0 < a < t.$$

Exercise 6. Using Taylor's expansion to write

$$u(a + da, t + dt) = u(a, t) + u_t(a, t)da + u_t(a, t)dt + \text{ higher order terms.}$$

5.2 Traveling Wave Fronts

Exercise 1. The traveling wave equation can be written

$$-cU' = DU'' - \frac{1}{2}(U^2)'.$$

Integrating, we get

$$-cU = DU' - \frac{1}{2}U^2 + A.$$

Using the boundary condition at $z = +\infty$ forces $A = 0$. Using the boundary condition at $z = -\infty$ gives the wave speed $c = 1/2$. Therefore

$$DU' = \frac{1}{2}U(U-1).$$

This DE has equilibria at $U = 0, 1$; the solution can be found by separating variables or noting it is a Bernoulli equation (see the Appendix on Differential Equations). The graph falls from left to right (decreasing), approaching 0 at plus infinity and 1 at minus infinity.

Exercise 2. The traveling wave equation may be written

$$-cU' = U'' - \frac{1}{3}(U^3)'.$$

Integrating, we find that the constant of integration is zero from the $z = +\infty$ boundary condition. Then

$$-cU = U' - \frac{1}{3}U^3.$$

Applying the condition at $z = -\infty$ we get

$$-cU_l = -\frac{1}{3}U_l^3,$$

or

$$-cU_l + \frac{1}{3}U_l^3 = -\frac{1}{3}U_l(3c - U_l^2) = 0.$$

Therefore $U_l = \sqrt{3c}$.

Exercise 3. To have constant states at infinity we must have $F(0, v_r) = 0 = F(u_l, 0) = 0$. The traveling wave equations are

$$-cU' = DU'' - \gamma U' - aF(U,V),$$
$$-cV' = -bF(U,V).$$

Clearly we may write a single equation

$$-cU' = DU'' - \gamma U' - \frac{ac}{b}V'.$$

Now we may integrate to get

$$-cU = DU' - \gamma U - \frac{ac}{b}V + A.$$

The right boundary condition forces $A = acv_r/b$. The left boundary condition then gives

$$-cu_l = -\gamma u_l + \frac{ac}{b}v_r$$

or

$$(\gamma - c)u_l = \frac{ac}{b}v_r > 0.$$

Therefore $c < \gamma$.

Exercise 5. The traveling wave equation is

$$-c\left[(1+b)U - mU^2\right]' = U'' - U'.$$

Integrating gives

$$-c\left[(1+b)U - mU^2\right] = U' - U + A.$$

¿From the boundary condition $U(+\infty) = 0$ we get $A = 0$. Since $U(-\infty) = 1$, we get

$$c = \frac{1}{1+b-m} > 0.$$

The differential equation then simplifies to

$$U' = (1 - c - cb)U + cmU^2,$$

which is a Bernoulli equation. It is also separable.

5.3 Equilibria and Stability

Exercise 1. The equilibria are roots of

$$f(u) = ru(1 - u/K) - hu = u(r - \frac{r}{K}u - h) = 0.$$

So the equilibria are

$$u_1 = 0, \quad u_2 = \frac{r-h}{r}K.$$

To check stability we calculate $f'(u_1) = r - \frac{2r}{K}u - h$. Then $f'(0) = r > 0$, so $u_1 = 0$ is unstable. Next

$$f'(\frac{r-h}{r}K) = r - \frac{2r}{K}\frac{r-h}{r}K - h = r - 2r + 2h - h = -r + h.$$

Therefore u_2 is stable if $r > h$ and unstable if $r < h$.

Exercise 2. Follow the steps indicated in the problem.

Exercise 3. The steady-state solution is $u_e = 3$, $v_e = 2/9$. The Jacobian matrix is

$$J = \begin{pmatrix} \frac{1}{3} & 9 \\ -\frac{4}{3} & -9 \end{pmatrix}.$$

Therefore, equation (5.36) becomes

$$\det\left(\begin{pmatrix} \sigma_n & 0 \\ 0 & \sigma_n \end{pmatrix} + n^2 \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{3} & 9 \\ -\frac{4}{3} & -9 \end{pmatrix}\right) = 0.$$

Simplifying gives

$$\det\begin{pmatrix} \sigma_n + n^2 D - \frac{1}{3} & -9 \\ -\frac{1}{3} & \sigma_n + n^2 + 9 \end{pmatrix} = 0.$$

Exercise 4. To obtain (a) just substitute $u_e(x)$ into the PDE and check the boundary conditions. To get (b) substitute $u = u_e(x) + U(x,t)$ into the PDE to obtain

$$U_t = u_e'' + U_{xx} + (u_e(x) + U(x,t))(1 - u_e(x) - U(x,t)),$$

or

$$U_t = u_e'' + U_{xx} + u_e(x)(1 - u_e(x)) - u_e(x)U + U(1 - u_e(x)) - U^2(x,t).$$

But $u_e'' + u_e(x)(1 - u_e(x)) = 0$, and neglecting the nonlinear term gives

$$U_t = U_{xx} + (1 - 2u_e(x))U,$$

which is the linearized perturbation equation. The boundary conditions are $U(\pm\pi/2) = 0$. For part (c) assume that $U = e^{\sigma t}g(x)$ and substitute to get

$$\sigma g = g'' + (1 - 2u_e(x))g,$$

or

$$g'' + \frac{\cos x - 5}{1 + \cos x}g = \sigma g,$$

with $g = 0$ at $x = \pm\pi/2$. Finally, to prove (d), we proceed as in the hint. If this BVP has a nontrivial solution, then it must be, say, positive somewhere in the interval. (The negative case can be treated similarly). So it must have a positive maximum in the interval. At this maximum, $g > 0$, $g'' < 0$. Therefore

$$\frac{\cos x - 5}{1 + \cos x} g < 0.$$

So the left side of the DE is negative, so $\sigma < 0$.