Sample Test 1 (p 265) Solutions

1. The characteristic equation associated with $u'' + 3u' - 10u = 0$ is $m^2 + 3m - 10 = 0$, which has roots $m = 2, -5$. Therefore $u(t) = ae^{2t} + be^{-5t}$.

2. This is a pure time equation, so integrate to get $u(t) = \int(\frac{1}{t} + t)dt + C = \ln t + \frac{1}{2}t^2 + C$. Substituting $t = 1$, $u = 0$ gives $C = -\frac{1}{2}$. Therefore $u(t) = \ln t + \frac{1}{2}t^2 - \frac{1}{2}$.

3. The DE is $2x'' + 3x = 0$ with initial conditions $x(0) = 0.25$ and $x'(0) = 1$. The general solution to the DE is $x(t) = a\cos\sqrt{\frac{3}{2}}t + b\sin\sqrt{\frac{3}{2}}t$. Then $x(0) = a = 0.25$ and $x'(0) = \sqrt{\frac{3}{2}}b = 1$, giving $b = \sqrt{\frac{2}{3}}$. Thus the amplitude is $A = \sqrt{a^2 + b^2} = \sqrt{\frac{35}{48}}$.

4. From Newton's second law, the DE is $v' = 3 - v$. Initially, $v(0) = 1$. By separating variables the solution is $v(t) = 3 - 2e^{-t}$. Setting $v = 2$ gives $t = \ln 2$.

5. Using the product rule for derivatives,
$$\frac{d}{dt}\left(t^2\int_1^t \frac{1}{r}e^{-r}dr\right) = t^2 \cdot \frac{1}{t}e^{-t} + 2t \cdot \int_1^t \frac{1}{r}e^{-r}dr.$$

6. TBA

7. We have $u' = t^2 - u = f(t, u)$. We know $u(-2) = 0$, and $h = 0.25$. By Euler's method,
$$\begin{aligned} u(-1.75) &= u(-2) + hf(-2, u(-2)) = 0 + 0.25((-2)^2 - 0) = 1.00, \\ u(-1.5) &= u(-1.75) + hf(-1.75, u(-1.75)) = 1.00 + 0.25((-1.75)^2 - 1.00) = 1.13, \end{aligned}$$
and so on.

8. Setting $t^2 - u = -1$ gives $u = t^2 + 1$, which is a standard, concave up parabola passing through $(0, 1)$.

9. The DE is $T' = -h(T - 68)$ with initial condition $T(0) = 85$. We take t=0 to be the time the body was discovered (noon). Either by separation of variables or using the formula on p. 46, we get $T(t) = 68 + 17e^{-ht}$. Now, $T(2) = 68 + 17e^{-2h} = 74$, which gives $h = 0.52$. Then $T(t) = 68 + 17e^{-0.52t}$. In this equation set $T = 98.6$ and solve for t. We get $t = -.1.13$ hours or -1 hour, 8 minutes. Thus the time of death is approximately 10:52 am.

Sample Test 2 (p 266) Solutions

1. Upon dividing the equations we can write

$$\frac{dy}{dx} = \frac{2}{x}.$$

Then $dy = (2/x)dx$, which integrates to $y = 2\ln x + C = \ln(C_1 x^2)$.

2. TBA

3. TBA

4. The homogeneous equation is $u' = \frac{3}{t}$, which has general solution $u_h(t) = Ct^3$. Assume $u(t) = C(t)t^3$ and substitute into the nonhomogeneous DE to get $C'(t) = 1/t^2$. Integrating gives $C(t) = -\frac{1}{3t^3} + K$. Therefore $u(t) = C(t)t^3 = \left(-\frac{1}{3t^3} + K\right)t^3 = -\frac{1}{3} + Kt^3$.

5. We have $u' = f(u) = -(u-2)(u-4)^2$. Setting $f(u) = 0$ gives the equilibria $u = 2, 4$. To check stability we compute $f'(u) = -2(u-2)(u-4) - (u-4)^2$. Then $f'(2) = -4 < 0$, so $u = 2$ is asymptotically stable; $f'(4) = 0$, so there is no information. But $f''(4) < 0$, showing the graph is concave down at $u = 4$. Thus $u = 4$ is semistable. Or, one could graph $f(u)$ to obtain these rusults.

6. Multiplying by t^2 gives the Cauchy-Euler equation $t^2u'' - tu' + 2u = 0$. The characteristic equation is $m(m-1) - m + 2 = 0$, which has roots $m = 1 \pm i$. Then the general solution is

$$u(t) = at\cos(\ln t) + bt\sin(\ln t).$$

7. TBA

8. By Newton's second law the DE is $2u'' = -au$. The force is $F(u) = -au$, so the potential energy is $V(u) = -\int F(u)du = \frac{1}{2}au^2$. The total energy is the kinetic energy plus the potential energy, or

$$\text{total energy} = \frac{1}{2}mu'^2 + \frac{1}{2}au^2.$$

Sample Test 3 (p267) Solutions

1. Dividing the equations gives $dy/dx = (2x-2)/4y$. Separating variables, $2y = (x-1)dx$, which integrates to $y^2 = \frac{1}{2}x^2 - x + C$.

2. Factoring out a p, we have $p' = p\left(\sqrt{2} - \frac{4p}{1+p^2}\right)$.So, the equilibrium solutions are $p = 0$ and the roots of $\sqrt{2} - \frac{4p}{1+p^2} = 0$. This last equation is a quadratic, $p^2 - \frac{4}{\sqrt{2}}p + 1 = 0$. By the quadratic formula we get $p = \frac{4}{\sqrt{2}} \pm 1$.

Sample Final Examination (p 268)

1. The characteristic equation is $m^2 - m - \frac{1}{2} = 0$, which has roots $m_\pm = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$. The general solution is $u(t) = ae^{m_+ t} + be^{m_- t}$.

2. The solution to the homogeneous equation is $u_h(t) = e^{-4t}(a + bt)$. To find the particular solution, guess a quadratic $u_p(t) = At^2 + Bt + C$.

3. Write the DE as
$$\frac{du}{dt} = \frac{1+t}{t}\frac{1}{3u^2+1},$$
which is separable. Then
$$\int (3u^2+1)du = \int \frac{1+t}{t}dt + C,$$
or
$$u^3 + u = \ln t + t + C.$$
Using $u = 1$ when $t = 1$ gives $C = 1$.

4. We have $u' = -u(u-2)^2 = f(u)$. The equilibrium are $u = 0, 2$. The derivative of f is $f'(u) = -(u-2)(1+2u)$. Because $f'(0) = 2 > 0$, we see $u = 0$ is unstable. Since $f'(0) = 0$ there is no information. But $f''(2) < 0$, and so there is a local maximum at $u = 2$, showing $u = 2$ is semistable.

5. TBA

6. The equation for an RC circuit is $Rq' + V = b(t)$, where V is the voltage across the capacitor. We know $q' = CV$, and so the DE and intial condition are
$$RCV' + V = b(t), \quad V(0) = 2.$$
Using $R = 1$ and $C = 2$ gives
$$V' + \frac{1}{2}V = \frac{1}{2}b(t).$$
This is a nonhomogeneous first order equation. The homogeneous solution is $V_h(t) = Ce^{-t/2}$. Therefore, assume $V(t) = C(t)e^{-t/2}$. Substituting gives, after simplification,
$$C'(t) = \frac{1}{2}b(t)e^{t/2}.$$
Then
$$C(t) = \frac{1}{2}\int_0^t b(s)e^{s/2}ds + K.$$
Therefore,
$$V(t) = e^{-t/2}\left(\frac{1}{2}\int_0^t b(s)e^{s/2}ds + K\right).$$
Applying the initial condition gives $V(0) = K = 2$.

7. TBA
8. TBA
9. TBA
10. Because $u' = u(\lambda^2 - u^2)$, the equilibrium solutions in the λu plane are the three straight lines $u = 0$, $u = \lambda$, $u = -\lambda$. We can check stability by checking the sign of $f'(u) = \lambda^2 - 3u^2$. We have $f'(0) = \lambda^2 > 0$, so $u = 0$ is unstable. Then $f'(\pm\lambda) = -2\lambda^2 < 0$, so $u = \lambda$ and $u = -\lambda$ are stable.
11. TBA
12. (a) Newton's second law gives $x'' = x^2(1 - x)$.