SOLUTIONS TO SELECTED EXERCISES

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CHAPTER 2

Partial Differential Equations on Unbounded Domains

1. Cauchy Problem for the Heat Equation

Exercise 1a. Making the transformation $r = (x-y)/\sqrt{4kt}$ we have

$$\begin{aligned} u(x,t) &= \int_{-1}^{1} \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt} dy \\ &= -\int_{(x+1)/\sqrt{4kt}}^{(x-1)/\sqrt{4kt}} \frac{1}{\sqrt{\pi}} e^{-r^2} dr \\ &= \frac{1}{2}\left(erf\ \left((x+1)/\sqrt{4kt}\right) - erf\ \left((x-1)/\sqrt{4kt}\right)\right) \end{aligned}$$

Exercise 1b. We have

$$u(x,t) = \int_0^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/4kt} e^{-y} dy$$

Now complete the square in the exponent of e and write it as

$$\begin{aligned} -\frac{(x-y)^2}{4kt} - y &= -\frac{x^2 - 2xy + y^2 + 4kty}{4kt} \\ &= -\frac{y + 2kt - x)^2}{4kt} + kt - x \end{aligned}$$

Then make the substitution in the integral

$$r = \frac{y + 2kt - x}{\sqrt{4kt}}$$

Then

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{(2kt-x)/\sqrt{4kt}}^{\infty} e^{-r^2} dr \\ &= \frac{1}{2} e^{kt-x} \left(1 - erf\left((2kt-x)/\sqrt{4kt}\right)\right) \end{aligned}$$

Exercise 2. We have

$$|u(x,t)| \le \int_R |G(x-y,t)||\phi(y)|dy \le M \int_R G(x-y,t)dy = M$$

Exercise 3. Use

$$\begin{aligned} erf(z) &= \frac{2}{\sqrt{\pi}}\int_0^z e^{-r^2}dr \\ &= \frac{2}{\sqrt{\pi}}\int_0^z (1-r^2+\cdots)dr \\ &= \frac{2}{\sqrt{\pi}}(z-\frac{z^3}{3}+\cdots) \end{aligned}$$

This gives

$$w(x_0,t)=\frac{1}{2}+\frac{x_0}{\pi\sqrt{t}}+\cdots$$

Exercise 4. The verification is straightforward. We guess the Green's function in two dimensions to be

$$\begin{aligned} g(x,y,t) &= G(x,t)G(y,t) \\ &= \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt}\frac{1}{\sqrt{4\pi kt}}e^{-y^2/4kt} \\ &= \frac{1}{4\pi kt}e^{-(x^2+y^2)/4kt} \end{aligned}$$

where G is the Green's function in one dimension. Thus g is the temperature distribution caused by a point source at $(x,y)=(0,0)$ at $t=0$. This guess gives the correct expression. Then, by superposition, we have the solution

$$u(x,y,t)=\int_{R^2}\frac{1}{4\pi kt}e^{-((x-\xi)^2+(y-\eta)^2)/4kt}\psi(\xi,\eta)d\xi d\eta$$

Exercise 6. Using the substitution $r=x/\sqrt{4kt}$ we get

$$\int_R G(x,t)dx=\frac{1}{\sqrt{\pi}}\int_R e^{-r^2}dr=1$$

Exercise 7. Verification is straightforward. The result does not contradict the theorem because the initial condition is not bounded.

2. Cauchy Problem for the Wave Equation

Exercise 1. Applying the initial conditions to the general solution gives the two equations

$$F(x)+G(x)=f(x), \quad -cF'(x)+cG'(x)=g(x)$$

We must solve these to determine the arbitrary functions F and G. Integrate the second equation to get

$$-cF(x)+cG(x)=\int_0^x g(s)ds+C$$

Now we have two linear equations for F and G that we can solve simultaneously.

Exercise 2. Using d'Alembert's formula we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct}\frac{ds}{1+0.25s^2} \\ &= \frac{1}{2c}2\arctan(s/2)\mid_{x-ct}^{x+ct} \\ &= \frac{1}{c}(\arctan((x+ct)/2)-\arctan((x-ct)/2)) \end{aligned}$$

Exercise 4. Let $u = F(x-ct)$. Then $u_x(0,t) = F'(-ct) = s(t)$. Then

$$F(t) = \int_0^t s(-r/c)dr + K$$

Then

$$u(x,t) = -\frac{1}{c}\int_0^{t-x/c} s(y)dy + K$$

Exercise 5. Letting $u = U/\rho$ we have

$$u_{tt} = U_{tt}/\rho, \quad u_\rho = U_\rho/\rho - U/\rho^2$$

and

$$u_{\rho\rho} = U_{\rho\rho}/\rho - 2U_\rho/\rho^2 + 2U/\rho^3$$

Substituting these quantities into the wave equation gives

$$U_{tt} = c^2 U_{\rho\rho}$$

which is the ordinary wave equation with general solution

$$U(\rho,t) = F(\rho-ct) + G(\rho+ct)$$

Then

$$u(\rho,t) = \frac{1}{r}(F(\rho-ct)+G(\rho+ct))$$

As a spherical wave propagates outward in space its energy is spread out over a larger volume, and therefore it seems reasonable that its amplitude decreases.

Exercise 6. The exact solution is, by d'Alembert's formula,

$$u(x,t) = \frac{1}{2}(e^{-|x-ct|}+e^{-|x+ct|}) + \frac{1}{2c}(\sin(x+ct)-\sin(x-ct))$$

Exercise 7. Use the fact that u is has the same value along a characteristic.

Exercise 8. Write

$$v = \int_R H(s,t)u(x,s)ds$$

where

$$h(s,t) = \frac{1}{\sqrt{4\pi t(k/c^2)}}e^{-s^2/(4t(k/c^2))}$$

which is the heat kernel with k replaced by k/c^2. Thus H satisfies

$$H_t - \frac{k}{c^2}H_{xx} = 0$$

Then, we have

$$\begin{aligned} v_t - kv_{xx} &= \int_R (H_t(s,t)u(x,s) - kH(s,t)u_{xx}(x,s))ds \\ &= \int_R (H_t(s,t)u(x,s) - (k/c^2)H(s,t)u_{ss}(x,s))ds \end{aligned}$$

where, in the last step, we used the fact that u satisfies the wave equation. Now integrate the second term in the last expression by parts twice. The generated boundary terms will vanish since H and H_s go to zero as $|s| \to \infty$. Then we get

$$v_t - kv_{xx} = \int_R (H_t(s,t)u(x,s) - (k/c^2)H_{ss}(s,t)u(x,s))ds = 0$$

3. Well-Posed Problems

Exercise 1. Consider the two problems

$$\begin{aligned} u_t + u_{xx} &= 0, \quad x \in R,\ t > 0 \\ u(x,0) &= f(x), \quad x \in R \end{aligned}$$

If $f(x) = 1$ the solution is $u(x,t) = 1$. If $f(x) = 1 + n^{-1}\sin nx$, which is a small change in initial data, then the solution is

$$u(x,t) = 1 + \frac{1}{n}e^{n^2 t}\sin nx$$

which is a large change in the solution. So the solution does not depend continuously on the initial data.

Exercise 2. Integrating twice, the general solution to $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y)$$

where F and G are arbitrary functions. Note that the equation is hyperbolic and therefore we expect the problem to be an evolution problem where data is carried forward from one boundary to another; so a boundary value problem should not be well-posed since the boundary data may be incompatible. To observe this, note that

$$u(x,0) = F(x) + G(0) = f(x). \quad u(x,1) = F(x) + G(1) = g(x)$$

where f and g are data imposed along $y = 0$ and $y = 1$, respectively. But these last equations imply that f and g differ by a constant, which may not be true.

Exercise 3. We subtract the two solutions given by d'Alembert's formula, take the absolute value, and use the triangle inequality to get

$$\begin{aligned}
|u^1 - u^2| &\leq \frac{1}{2}|f^1(x-ct) - f^2(x-ct)| + \frac{1}{2}|f^1(x+ct) - f^2(x+ct)| \\
&\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} |g^1(s) - g^2(s)|ds \\
&\leq \frac{1}{2}\delta_1 + \frac{1}{2}\delta_1 + \frac{1}{2c}\int_{x-ct}^{x+ct} \delta_2 ds \\
&= \delta_1 + \frac{1}{2c}\delta_2(2ct) \\
&\leq \delta_1 + T\delta_2
\end{aligned}$$

4. Semi-Infinite Domains

Exercise 2. We have

$$u(x,t) = \int_0^\infty (G(x-y,t) - G(x+y,t))dy = erf\ (x/\sqrt{4kt})$$

Exercise 3. For $x > ct$ we use d'Alembert's formula to get

$$u(x,t) = \frac{1}{2}((x-ct)e^{-(x-ct)} + (x+ct)e^{-(x+ct)})$$

For $0 < x < ct$ we have from (2.29) in the text

$$u(x,t) = \frac{1}{2}((x+ct)e^{-(x+ct)} - (ct-x)e^{-(ct-x)})$$

Exercise 4. Letting $w(x,t) = u(x,t) - 1$ we get the problem

$$w_t = kw_{xx}, \quad w(0,t) = 0, \ \ t > 0, \ \ ; w(x,0) = -1, \ \ x > 0$$

Now we can apply the result of the text to get

$$w(x,t) = \int_0^\infty (G(x-y,t) - G(x+y,t))(-1)dy = -erf\ (x/\sqrt{4kt})$$

Then

$$u(x,t) = 1 - erf\ (x/\sqrt{4kt})$$

Exercise 5. The problem is

$$\begin{aligned}
u_t = ku_{xx}, \ \ x > 0, \ t > 0 \\
u(x,0) = 7000, \ \ x > 0 \\
u(0,t) = 0, \ \ t > 0
\end{aligned}$$

From Exercise 2 we know the temperature is

$$u(x,t) = 7000\ erf\ (x/\sqrt{4kt}) = 7000\frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-r^2}dr$$

The geothermal gradient at the current time t_c is

$$u_x(0,t_c) = \frac{7000}{\sqrt{\pi k t_c}} = 3.7 \times 10^{-4}$$

Solving for t gives

$$t_c = 1.624 \times 10^{16} \text{ sec } = 5.15 \times 10^8 \text{ yrs}$$

This gives a very low estimate; the age of the earth is thought to be about 15 billion years.

There are many ways to estimate the amount of heat lost. One method is as follows. At $t=0$ the total amount of heat was

$$\int_S \rho c u \, dV = 7000\rho c \frac{4}{3}\pi R^3 = 29321\rho c R^3$$

where S is the sphere of radius $R = 4000$ miles and density ρ and specific heat c. The amount of heat leaked out can be calculated by integrating the geothermal gradient up to the present day t_c. Thus, the amount leaked out is approximately

$$\begin{aligned}(4\pi R^2)\int_0^{t_c} -K u_x(0,t)dt &= -4\pi R^2 \rho c k(7000)\int_0^{t_c} \frac{1}{\sqrt{\pi k t}}dt \\ &= -\rho c R^2 (1.06 \times 10^{12})\end{aligned}$$

So the ratio of the heat lost to the total heat is

$$\frac{\rho c R^2(1.06 \times 10^{12})}{29321\rho c R^3} = \frac{3.62 \times 10^7}{R} = 5.6\%$$

Exercise 6. Follow the suggested steps. **Exercise 7.** The left side of the equation

is the flux through the surface. The first term on the right is Newton's law of cooling and the second term is the radiation heating.

5. Sources and Duhamel's Principle

Exercise 1. The solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \sin s ds \\ &= \frac{1}{c^2}\sin x - \frac{1}{2c^2}(\sin(x-ct) + \sin(x+ct)) \end{aligned}$$

Exercise 2. The solution is

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-y, t-\tau) \sin y \, dy d\tau$$

where G is the heat kernel.

Exercise 3. The problem

$$w_t(x,t,\tau) + c w_x(x,t,\tau) = 0, \quad w(x,0,\tau) = f(x,\tau)$$

has solution (see Chapter 1)

$$w(x,t,\tau) = f(x-ct,\tau)$$

Therefore, by Duhamel's principle, the solution to the original problem is

$$u(x,t) = \int_0^t f(x-c(t-\tau),\tau)d\tau$$

Applying this formula when $f(x,t) = xe^{-t}$ and $c = 2$ gives

$$u(x,t) = \int_0^t (x-2(t-\tau))e^{-\tau}d\tau$$

This integral can be calculated using integration by parts or a computer algebra program. We get

$$u(x,t) = -(x-2t)(e^{-t}-1) - 2te^{-t} + 2(1-e^{-t})$$

6. Laplace Transforms

Exercise 4. Using integration by parts, we have

$$\begin{aligned} L\left(\int_0^t f(\tau)d\tau\right) &= \int_0^\infty \left(\int_0^t f(\tau)d\tau\right)e^{-st}dt \\ &= -\frac{1}{s}\int_0^\infty \left(\int_0^t f(\tau)d\tau\right)\frac{d}{ds}e^{-st}dt \\ &= -\frac{1}{s}\int_0^t f(\tau)d\tau \cdot e^{-st}\,|_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt \\ &= \frac{1}{s}F(s) \end{aligned}$$

Exercise 5. Since $H = 0$ for $x < a$ we have

$$\begin{aligned} L\left(H(t-a)f(t-a)\right) &= \int_a^\infty f(t-a)e^{-st}dt \\ &= \int_0^\infty f(\tau)e^{-s(\tau+a)}d\tau = e^{-as}F(s) \end{aligned}$$

where we used the substitution $\tau = t-a,\ d\tau = dt$.

Exercise 8. The model is

$$\begin{aligned} u_t = u_{xx}, &\quad x>0,\ t>0 \\ u(x,0) = u_0, &\quad x>0 \\ -u_x(0,t) = -u(0,t) & \end{aligned}$$

Taking the Laplace transform of the PDE we get

$$U_{xx} - sU = -u_0$$

The bounded solution is

$$U(x,s) = a(s)e^{-x\sqrt{s}} + \frac{u_0}{s}$$

The radiation boundary condition gives

$$-a(s)\sqrt{s} = a(s) + \frac{u_0}{s}$$

or

$$a(s) = -\frac{u_0}{s(1+\sqrt{s})}$$

Therefore, in the transform domain

$$U(x,s) = -\frac{u_0}{s(1+\sqrt{s})}e^{-x\sqrt{s}} + \frac{u_0}{s}$$

Using a table of Laplace transforms we find

$$u(x,t) = u_0 - u_0\left[erfc\left(\frac{x}{\sqrt{4t}}\right) - erfc\left(\sqrt{t} + \frac{x}{\sqrt{4t}}\right)e^{x+t}\right]$$

where $erfc(z) = 1 - erf(z)$.

Exercise 10. Taking the Laplace transform of the PDE gives, using the initial conditions,

$$U_{xx} - \frac{s^2}{c^2}U = -\frac{g}{sc^2}$$

The general solution is

$$U(x,s) = A(s)e^{-sx/c} + B(s)e^{sx/c} + \frac{g}{s^3}$$

To maintain boundedness, set $B(s) = 0$. Now $U(0,s) = 0$ gives $A(s) = -g/s^3$. Thus

$$U(x,s) = -\frac{g}{s^3}e^{-sx/c} + \frac{g}{s^3}$$

is the solution in the transform domain. Now, from a table or computer algebra program,

$$L^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2}, \quad L^{-1}(F(s)e^{-as}) = H(t-a)f(t-a)$$

Therefore

$$L^{-1}\left(\frac{1}{s^3}e^{-xs/c}\right) = H(t-x/c)\frac{(t-x/c)^2}{2}$$

Hence

$$u(x,t) = \frac{gt^2}{2} - gH(t-x/c)\frac{(t-x/c)^2}{2}$$

Exercise 11. Taking the Laplace transform of the PDE while using the initial condition gives, for $U = U(x,y,s)$,

$$U_{yy} - pU = 0$$

The bounded solution of this equation is

$$U = a(x,s)e^{-y\sqrt{s}}$$

The boundary condition at $y = 0$ gives $sU(x,o,s) = -U_x(x,0,s)$ or $a = -a_x$, or

$$a(x,s) = f(s)e^{-xs}$$

The boundary condition at $x = u = 0$ forces $f(s) = 1/s$. Therefore

$$U(x,y,s) = \frac{1}{s}e^{-xs}e^{-y\sqrt{s}}$$

From the table of transforms

$$u(x,y,t) = 1 - erf((y-x)/\sqrt{4t})$$

Exercise 13. Taking the Laplace transform of the PDE gives, using the initial conditions,

$$U_{xx} - \frac{s^2}{c^2}U = 0$$

The general solution is

$$U(x,s) = A(s)e^{-sx/c} + B(s)e^{sx/c}$$

To maintain boundedness, set $B(s) = 0$. Now The boundary condition at $x = 0$ gives $U(0,s) = G(s)$ which forces $A(s) = G(s)$. Thus

$$U(x,s) = G(s)e^{-sx/c}$$

Therefore, using Exercise 4, we get

$$u(x,t) = H(t - x/c)g(t - x/c)$$

7. Fourier Transforms

Exercise 1. The convolution is calculated from

$$x \star e^{-x^2} = \int_{-\infty}^{\infty} (y-x)e^{-y^2}dy$$

Exercise 2. From the definition we have

$$\begin{aligned}
\mathcal{F}^{-1}(e^{-a|\xi|})) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a|\xi|}e^{-ix\xi}d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{a\xi}e^{-ix\xi}d\xi + \frac{1}{2\pi}\int_{0}^{\infty} e^{-a\xi}e^{-ix\xi}d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{a\xi-ix\xi}d\xi + \frac{1}{2\pi}\int_{0}^{\infty} e^{-a\xi-ix\xi}d\xi \\
&= \frac{1}{2\pi}\frac{1}{a-ix}e^{(a-ix)\xi}\ |_{-\infty}^{0} + \frac{1}{2\pi}\frac{1}{-a-ix}e^{(-a-ix)\xi}\ |_{0}^{\infty} \\
&= \frac{a}{\pi}\frac{1}{a^2+x^2}
\end{aligned}$$

Exercise 3a. Using the definition of the Fourier transform

$$2\pi\mathcal{F}^{-1}(-\xi) = \int_{-\infty}^{\infty} u(x)e^{-i(-\xi)x}dx = \mathcal{F}(u)(\xi)$$

Exercise 3b. From the definition,

$$\begin{aligned}\hat{u}(\xi+a) &= \int_{-\infty}^{\infty} u(x)e^{i(\xi+a)x}dx \\ &= \int_{-\infty}^{\infty} u(x)e^{iax}e^{i\xi x}dx \\ &= \mathcal{F}(e^{iax}u)(\xi)\end{aligned}$$

Exercise 3c. Use 3(a) or, from the definition,

$$\mathcal{F}(u(x+a)) = \int_{-\infty}^{\infty} u(x+a)e^{i\xi x}dx = \int_{-\infty}^{\infty} u(y)e^{i\xi(y-a)}dy = e^{-ia\xi}\hat{u}(\xi)$$

Exercise 6. From the definition

$$\begin{aligned}\hat{u}(\xi) &= \int_{0}^{\infty} e^{-ax}e^{i\xi x}dx \\ &= \int_{0}^{\infty} e^{(i\xi-a)x}dx \\ &= \frac{1}{i\xi-a}e^{(i\xi-a)x}\ |_0^\infty \\ &= \frac{1}{a-i\xi}\end{aligned}$$

Exercise 7. Observe that

$$xe^{-ax^2} = -\frac{1}{2a}\frac{d}{dx}e^{-ax^2}$$

Then

$$\mathcal{F}(xe^{-ax^2}) = -\frac{1}{2a}(-i\xi)\mathcal{F}(e^{-ax^2}).$$

Exercise 9. Take transforms of the PDE to get

$$\hat{u}_t = (-i\xi)^2\hat{u} + \hat{f}(\xi,t)$$

Solving this as a linear, first order ODE in t with ξ as a parameter, we get

$$\hat{u}(\xi,t) = \int_0^t e^{-x^2(t-\tau)}\hat{f}(\xi,\tau)d\tau$$

Taking the inverse Fourier transform, interchanging the order of integration, and applying the convolution theorem gives

$$\begin{aligned}u(x,t) &= \int_0^t \mathcal{F}^{-1}\left[e^{-x^2(t-\tau)}\hat{f}(\xi,\tau)\right]d\tau \\ &= \int_0^t \mathcal{F}^{-1}\left[e^{-x^2(t-\tau)}\right]\star f(x,\tau)d\tau \\ &= \int_0^t\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi(t-\tau)}}e^{-(x-y)^2/4(t-\tau)}f(y,\tau)dyd\tau\end{aligned}$$

Exercise 10. Proceeding exactly in the same way as in the derivation of (2.65) in the text, but with k replaced by I, we obtain the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi it}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4it} f(y)dy$$

where $u(x,0) = f(x)$. Thus

$$u(x,t) = \frac{1}{\sqrt{4\pi it}} \int_{-\infty}^{\infty} e^{i(x-y)^2/4t - y^2} dy$$

Here, in the denominator, $\sqrt{i}$ denotes the root with the positive real part, that is $\sqrt{i} = (1+i)/\sqrt{2}$.

Exercise 11. Letting $v = u_y$ we have

$$v_{xx} + v_{yy} = 0, \quad x \in R,\ y > 0; \quad v(x,0) = g(x)$$

Hence

$$v(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yg(\tau)}{(x-\tau)^2 + y^2} d\tau$$

Then

$$\begin{aligned}
u(x,y) &= \int_0^y v(x,\xi)d\xi \\
&= \int_0^y \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yg(\tau)}{(x-\tau)^2 + y^2} d\tau d\xi \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^y \frac{y}{(x-\tau)^2 + y^2} d\xi d\tau \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\tau) \left(\ln\left((x-\tau)^2 + y^2\right) - \ln\left((x-\tau)^2\right)\right) d\tau \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x-\xi) \ln\left(\xi^2 + y^2\right) d\tau + C
\end{aligned}$$

Exercise 12. We have

$$\begin{aligned}
u(x,y) &= \frac{y}{\pi} \int_{-l}^{l} \frac{d\tau}{(x-\tau)^2 + y^2} \\
&= \frac{1}{\pi} \left(\arctan\left(\frac{l-x}{y}\right) + \arctan\left(\frac{l+x}{y}\right)\right)
\end{aligned}$$

Exercise 13. From the definition of the Fourier transform

$$\begin{aligned}
\mathcal{F}(u_x) &= \int_{-\infty}^{\infty} u_x(x,t) e^{i\xi x} dx \\
&= u(x,t)e^{i\xi x} \mid_{-\infty}^{\infty} - i\xi \int_{-\infty}^{\infty} u(\xi,t) e^{i\xi x} dx \\
&= -i\xi \hat{u}(\xi,t)
\end{aligned}$$

For the second derivative, integrate by parts twice and assume u and u_x tend to zero as $x \to \pm\infty$ to get rid of the boundary terms.

Exercise 14. In this case where f is a square wave signal,

$$\mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x)e^{i\xi x}dx = \int_{-a}^{a} e^{i\xi x}dx = \frac{2\sin \xi x}{\xi}$$

Exercise 15. Taking the Fourier transform of the PDE

$$u_t = Du_{xx} - cu_x$$

gives

$$\hat{u}_t = -(D\xi^2 + i\xi c)\hat{u}$$

which has general solution

$$\hat{u}(\xi, t) = C(\xi)e^{-D\xi^2 t - i\xi ct}$$

The initial condition forces $C(\xi) = \hat{\phi}(\xi)$ which gives

$$\hat{u}(\xi, t) = \hat{\phi}(\xi)e^{-D\xi^2 t - i\xi ct}$$

Using

$$\mathcal{F}^{-1}\left(e^{-D\xi^2 t}\right) = \frac{1}{\sqrt{4\pi Dt}}e^{-x^2/4Dt}$$

and

$$\mathcal{F}^{-1}\left(\hat{u}(\xi, t)e^{-ia\xi}\right) = u(x + a)$$

we have

$$\mathcal{F}^{-1}\left(e^{-i\xi ct}e^{-D\xi^2 t}\right) = \frac{1}{\sqrt{4\pi Dt}}e^{-(x+vt)^2/4Dt}$$

Then, by convolution,

$$u(x, t) = \phi \star \frac{1}{\sqrt{4\pi Dt}}e^{-(x+vt)^2/4Dt}$$

Exercise 16. (a) Substituting $u = \exp(i(kx - \omega t))$ into the PDE $u_t + u_{xxx} = 0$ gives $-i\omega + (ik)^3 = 0$ or $\omega = -k^3$. Thus we have solutions of the form

$$u(x, t) = e^{i(kx+k^3 t)} = e^{ik(x+k^2 t)}$$

The real part of a complex-valued solution is a real solution, so we have solutions of the form

$$u(x, t) = \cos[k(x + k^2 t)]$$

These are left traveling waves moving with speed k^2. So the temporal frequency ω as well as the wave speed $c = k^2$ depends on the spatial frequency, or wave number, k. Note that the wave length is proportional to $1/k$. Thus, higher frequency waves are propagated faster.

(b) Taking the Fourier transform of the PDE gives

$$\hat{u}_t = -(-i\xi)^3\hat{u}$$

This has solution

$$\hat{u}(\xi, t) = \hat{\phi}(\xi)e^{-i\xi^3 t}$$

where $\hat{\phi}$ is the transform of the initial data. By the convolution theorem,

$$u(x, t) = \phi(x) \star F^{-1}(e^{-i\xi^3 t})$$

To invert this transform we go to the definition of the inverse. We have

$$\begin{aligned}\mathcal{F}^{-1}(e^{-i\xi^3 t}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi^3 t}e^{-i\xi x}d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(\xi^3 t+\xi x)d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos\left(\frac{z^3}{3}+\frac{zx}{(3t)^{1/3}}\right)\frac{1}{(3t)^{1/3}}dz \\ &= \frac{1}{(3t)^{1/3}}\mathrm{Ai}\left(\frac{x}{(3t)^{1/3}}\right)\end{aligned}$$

where we made the substitution $\xi = z/(3t)^{1/3}$ to put the integrand in the form of that in the Airy function. Consequently we have

$$u(x,t) = \frac{1}{(3t)^{1/3}}\int_{-\infty}^{\infty}\phi(x-y)\mathrm{Ai}\left(\frac{y}{(3t)^{1/3}}\right)dy.$$

Exercise 18. The problem is

$$u_{tt} = c^2 u_{xx} = 0, \quad x \in R,\ t > 0$$
$$u(x,0) = f(x),\ \ u_t(x,0) = 0,\ \ x \in R$$

Taking Fourier transforms of the PDE yields

$$\hat{u}_{tt} + c^2\xi^2\hat{u} = 0$$

whose general solution is

$$\hat{u} = A(\xi)e^{i\xi ct} + B(\xi)e^{-i\xi ct}$$

From the initial conditions, $\hat{u}(\xi,0) = \hat{f}(\xi)$ and $\hat{u}_t(\xi,0) = 0$. Thus $A(\xi) = B(\xi) = 0.5\hat{f}(\xi)$. Therefore

$$\hat{u}(\xi,t) = 0.5\hat{f}(\xi)(e^{i\xi ct} + e^{-i\xi ct})$$

Now we use the fact that

$$F^{-1}\left(\hat{f}(\xi)e^{ia\xi}\right) = f(x-a)$$

to invert each term. Whence

$$u(x,t) = 0.5(f(x-ct) + f(x+ct))$$

Exercise 20. Notice the left side is a convolution. Take the transform of both sides, use the convolution theorem, and solve for $\hat{f}$. Then invert to get f.