

FOURIER SERIES

Gary H. Meisters

February 17, 1992

Here are some facts about Fourier Series — useful for pde and elsewhere. Proofs of Lemmas are easy exercises, and not given. On the other hand, proofs of LEMMAS are harder; their proofs are indicated, or a reference is given.

1 The L^1 - Theory

Consider functions $f : \mathbf{R} \rightarrow \mathbf{C}$ of period $a > 0$: $f(x + a) = f(x) \quad \forall x \in \mathbf{R}$.

Lemma 1. $\forall x \in \mathbf{R}$, $\int_x^{x+a} f(s) ds = \int_0^a f(s) ds$.

Definition 1.1. The (complex) FOURIER COEFFICIENTS (F. C.) of f :

$$\hat{f}(k) := \frac{1}{a} \int_0^a f(s) e^{-2\pi i k s / a} ds, \quad k \in \mathbf{Z}. \quad (1)$$

Definition 1.2. The FOURIER SERIES (F. S.) representing f is the series:

$$f(x) = \sum_{k \in \mathbf{Z}} \hat{f}(k) e^{2\pi i k x / a}, \quad k \in \mathbf{Z}. \quad (2)$$

Definition 1.3. The N^{th} (symmetric) PARTIAL SUM of the F. S. of f :

$$S_N(f)(x) := \sum_{|k| \leq N} \hat{f}(k) e^{2\pi i k x / a} = \frac{1}{a} \int_0^a f(s) D_N(x - s) ds, \quad (3)$$

where $D_N(\theta)$ denotes the DIRICHLET KERNEL given by

$$D_N(\theta) := \sum_{|k| \leq N} e^{2\pi i k \theta / a} = \frac{\sin(\pi(2N + 1)\theta/a)}{\sin(\pi\theta/a)}. \quad (4)$$

To obtain the second equality in 4, multiply $e^{-\pi i \theta / a}$ through the identity

$$(e^{2\pi i \theta / a} - 1) D_N(\theta) = e^{2\pi i (N+1)\theta/a} - e^{-2\pi i N\theta/a}.$$

Definition 1.4. The REAL FOURIER COEFFICIENTS of f are:

$$a_k = \frac{2}{a} \int_0^a f(t) \cos(2\pi kt/a) dt, \quad b_k = \frac{2}{a} \int_0^a f(t) \sin(2\pi kt/a) dt, \quad k \in \mathbf{N}. \quad (5)$$

Lemma 2. $\forall k \in \mathbf{N}$, $a_k = [\hat{f}(k) + \hat{f}(-k)]$, $b_k = i[\hat{f}(k) - \hat{f}(-k)]$, and

$$S_N(f)(x) = \frac{a_0}{2} + \sum_{k=1}^N \{a_k \cos(2\pi kx/a) + b_k \sin(2\pi kx/a)\}, \quad \forall N \in \mathbf{N}. \quad (6)$$

Lemma 3. The Fourier Coefficients of L^1 -functions are bounded:

$$|\hat{f}(k)| \leq \frac{1}{a} \int_0^a |f(s)| ds \quad \forall k \in \mathbf{Z}.$$

Lemma 4. The Orthogonality Relations:

$$\frac{1}{a} \int_0^a e^{2\pi iks/a} ds = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases} \quad (7)$$

Lemma 5. $\forall n \geq 1$, $\widehat{f^{(n)}}(k) = (2\pi ik/a)^n \hat{f}(k)$. (F.C.'s of the derivatives of f .)

LEMMA 6. If periodic f is continuous, then $\forall \epsilon > 0$ there is a trig polynomial

$$P(x) = \sum_{|k| \leq N} c_k e^{2\pi ikx/a} \quad \text{such that } |f(x) - P(x)| < \epsilon \quad \forall x \in \mathbf{R}.$$

Proof: This is one form of the Weierstrass Approximation Theorem.¹ Q.E.D.

LEMMA 7. (Uniqueness.) If f is continuous and $\hat{f}(k) = 0 \quad \forall k \in \mathbf{Z}$, then $f \equiv 0$.

Proof (when $a = 1$): Given $\epsilon > 0$, choose $P(x)$ as in *LEMMA 6*. Then

$$\int_0^1 |f(x)|^2 dx = \int_0^1 \overline{f(x)} f(x) dx - \int_0^1 \overline{f(x)} P(x) dx \leq \epsilon \sqrt{\int_0^1 |f(x)|^2 dx}.$$

Q.E.D.

Lemma 8. $D(\theta)$ is continuous, has period a , and $\frac{1}{a} \int_0^a D_N(\theta) d\theta = 1$.

Definition 1.5. The CONVOLUTION PRODUCT of two periodic functions:

$$(f * g)(x) := \int_0^1 f(x-s)g(s) ds = \int_0^1 f(t)g(x-t) dt. \quad (8)$$

Lemma 9. (The Exchange Formula) $\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k)$, $\forall k \in \mathbf{Z}$.

¹R. B. Burckel, Bishop's Stone-Weierstrass Theorem, *Amer. Math. Monthly* 91 (1984), 22–32.

LEMMA 10. (The Riemann-Lebesgue Lemma)

$$\text{If } f \in L^1[0, 1], \text{ then } \lim_{|k| \rightarrow \infty} \hat{f}(k) = 0.$$

Proof:

$$\hat{f}(n) = \int_0^1 e^{-2\pi i n x} f(x) dx,$$

and therefore also

$$-\hat{f}(n) = e^{i\pi} \hat{f}(n) = \int_0^1 e^{-2\pi i n u} f(u + \frac{1}{2n}) du.$$

Subtracting the last from the first we obtain

$$2\hat{f}(n) = \int_0^1 e^{-2\pi i n x} [f(x) - f(x + \frac{1}{2n})] dx.$$

And consequently,

$$|\hat{f}(n)| \leq \frac{1}{2} \int_0^1 |f(x) - f(x + \frac{1}{2n})| dx.$$

But the latter tends to zero, as $|n|$ tends to infinity, because L^1 -functions are continuous in the L^1 -norm. Q.E.D.

Theorem 1: (The Chernoff-Fourier Convergence Theorem.) Let f be integrable and suppose that f is differentiable at the point x_0 . Then the partial sums

$$S_{m,n}(x_0) \equiv \sum_{k=-m}^n \hat{f}(k) e^{2\pi i k x}$$

converge to $f(x_0)$ as $m, n \rightarrow \infty$.

Proof²: We may suppose that $x_0 = 0$ and $f(x_0) = 0$: Just subtract a constant from f and shift the origin. Since $f(0) = 0$ and $f'(0)$ exists, the function $g(x) = f(x)/[e^{2\pi i x} - 1]$ is bounded near 0 and thus is integrable because f is integrable. Then $f(x) = (e^{2\pi i x} - 1)g(x)$ so that the Fourier coefficients for f satisfy $\hat{f}(k) = \hat{g}(k-1) - \hat{g}(k)$. Thus the Fourier Series of f is a **telescoping** series. Indeed,

$$S_{m,n}(0) = \sum_{k=-m}^n \hat{f}(k) = \hat{g}(-m-1) - \hat{g}(n),$$

and this tends to 0 ($= f(0)$) by the Riemann-Lebesgue Lemma applied to g .

Q.E.D.

²Paul R. Chernoff, *Amer. Math. Monthly* (1980), 399-400.

2 Periodic Distributions

Definition 2.1. (Distributions on open subsets of \mathbf{R}^n , Laurent Schwartz c.1950.)³

A **distribution** u , on an open subset Ω of \mathbf{R}^n , is a linear form on the vector space $\mathcal{D}(\Omega)$ [of all complex-valued C^∞ -functions ϕ with compact $\text{supp } \phi \subset \Omega$], such that for every compact set $K \subset \Omega$, there exist constants C and k such that

$$|u(\phi)| \leq C \sum_{|\alpha| \leq k} \text{sup} |\partial^\alpha \phi|, \quad \forall \phi \in \mathcal{D}(\Omega) \text{ with } \text{supp } \phi \subset K. \quad (9)$$

The set of all distributions u on Ω is denoted by $\mathcal{D}'(\Omega)$. The (linear) action of a distribution u on a “test function” ϕ is also denoted by $\langle u, \phi \rangle \in \mathbf{C}$. Every function f locally integrable on Ω gives rise to a distribution on Ω (also denoted f) by virtue of the linear action defined by $\langle f, \phi \rangle := \int_\Omega f(x)\phi(x) dx$. The “Dirac delta function” $\langle \delta, \phi \rangle := \phi(0)$ is a distribution which does not correspond to any locally integrable function. Translation $\tau_x u = u_x$ of a distribution u by a vector $x \in \mathbf{R}^n$ is defined by the equation $\langle u_x, \phi \rangle := \langle u, \phi_{-x} \rangle$, where $\phi_x(y) := \phi(y - x)$. The (distributional) derivative u' of u is defined by $\langle u', \phi \rangle := -\langle u, \phi' \rangle$. The Fourier Transform (when it exists) of u is defined by the equations

$$\langle \hat{u}, \phi \rangle := \langle u, \hat{\phi} \rangle, \text{ where } \hat{\phi}(x) := \int_{-\infty}^{+\infty} \phi(y) e^{-2\pi i x \cdot y} dy.$$

Generally, any operation valid for C^∞ -functions ϕ can be defined for distributions in the manner illustrated above. For example, we say that a distribution u is periodic of period a if $u_a = u$. The space of all of these distributions invariant under translation by a is denoted by $(\mathcal{D}')_a$. However, the entire theory of periodic distributions can be developed in a direct and simpler fashion—without first learning about the general theory of distributions—as the dual space of the space of C^∞ -periodic functions. \square

So let us first restrict our attention to the simpler special case of all periodic distributions on the real line. (Set $n = 1$ and $\Omega = \mathbf{R}$). These can be defined independently of the above definition as follows.

Definition 2.2. (The vector space \mathcal{P}_a of C^∞ periodic test functions.)

Let \mathcal{P}_a denote the complex vector space of all complex-valued infinitely differentiable functions (on the real line \mathbf{R}), each of positive period a . This vector space \mathcal{P}_a , equipped with the sequence of norms

$$\|f\|_{(N)} := \sum_{k \leq N} \|f^{(k)}\|_\infty, \quad N = 1, 2, 3, \dots, \quad (10)$$

is a complete metrizable locally convex topological vector space (*Fréchet space*) which is dense in the Banach space \mathcal{C}_a of all continuous functions $f: \mathbf{R} \rightarrow \mathbf{C}$ of period a (equipped with the supremum norm $\|f\|_\infty$). For example, the C^∞ -functions $e^{2\pi i n x/a}$, $n \in \mathbf{Z}$, generate an algebra dense in \mathcal{C}_a . \square

³Lars Hörmander, *The Analysis of Linear Partial Differential Operators I Distribution Theory and Fourier Analysis*, Springer-Verlag 1983.

Definition 2.3. (The space of all periodic distributions on the real line \mathbf{R} as the dual of \mathcal{P}_a .)

A **periodic distribution** S of period a on the real line \mathbf{R} is a continuous linear form on the Fréchet space \mathcal{P}_a . A linear form S on \mathcal{P}_a (i.e., an element of the algebraic dual $(\mathcal{P}_a)^\#$) is denoted either by the usual function notation $f \rightarrow S(f)$, or by the square-bracket notation $f \rightarrow [S, f]$. Such a linear form S is continuous iff for some integer $N \geq 0$ there exists a constant $C_N > 0$ such that

$$|S(f)| \leq C_N \|f\|_{(N)} \quad \forall f \in \mathcal{P}_a. \quad (11)$$

The space of all distributions S of period a on \mathbf{R} is denoted by $(\mathcal{P}_a)'$. The **order** of S is the least integer $N \geq 0$ for which such a constant C_N exists. \square

An element S in $(\mathcal{P}_a)'$ has a (unique) extension S_0 to a continuous linear functional on the Banach space \mathcal{C}_a iff S has order 0. By the Kakutani-Riesz Representation Theorem, to each $S_0 \in (\mathcal{C}_a)'$ there corresponds a unique (complex-valued) Radon measure μ on $[0, a]$ such that for all $f \in \mathcal{C}_a$

$$[S_0, f] = \frac{1}{a} \int_0^a f(x) d\mu(x).$$

Consequently, $(\mathcal{C}_a)'$ can be identified with the space of complex Radon measures on $[0, a]$; and so $S \in (\mathcal{P}_a)'$ can be identified with a complex Radon measure μ iff S has order 0. In this case we may write $S = S_0 = \mu$. The (distributional) derivative S' of S is defined by $[S', f] := -[S, f']$. Each function $g \in L^1[0, a]$ defines a periodic distribution on \mathbf{R} (also denoted g) by virtue of the linear action $[g, f] := \frac{1}{a} \int_0^a g(x)f(x)dx$, $\forall f \in \mathcal{P}_a$. Thus we may write $L^1[0, a] \subset (\mathcal{P}_a)'$.

The action of a periodic distribution S on a periodic “test function” f is denoted by $[S, f]$ in order to distinguish it from $\langle S, \phi \rangle$ which can also make sense but be different. For example, the **Poisson distribution** P_a (of period a), also called “a row of deltas”, is defined by the formula $[P_a, f] := f(0)$, $\forall f \in \mathcal{P}_a$. Thus $[P_a, f] = \langle \delta, f \rangle = f(0)$, $\forall f \in \mathcal{P}_a$, because distributions of compact support, such as δ , extend (as continuous linear forms) to the space $\mathcal{E}(\mathbf{R})$ of all C^∞ -functions (with no restriction on their supports). Note that for each a , $\mathcal{P}_a \subset \mathcal{E}(\mathbf{R})$. By the way, the topological dual $\mathcal{E}'(\mathbf{R})$ of $\mathcal{E}(\mathbf{R})$, i.e., the space of all continuous linear forms on $\mathcal{E}(\mathbf{R})$, consists of those distributions $u \in \mathcal{D}'(\mathbf{R})$ which have *compact* supports in \mathbf{R} . The Fourier Coefficients of any $S \in (\mathcal{P}_a)'$ are given by $\hat{S}(n) := [S, e^{-2\pi i n x/a}]$, $\forall n \in \mathbf{Z}$. Consequently, $\widehat{P_a}(n) := [P_a, e^{-2\pi i n x/a}] = 1$, so

$$P_a = \llbracket x/a \rrbracket' = \sum_{n=-\infty}^{+\infty} \delta(x - na) = \sum_{n=-\infty}^{+\infty} e^{2\pi i n x/a}. \quad (12)$$

This yields versions of the classical **Poisson summation formula** such as

$$\langle P_a, \phi \rangle = \sum_{n=-\infty}^{+\infty} \phi(na) = \sum_{n=-\infty}^{+\infty} \hat{\phi}(n/a), \quad \forall \phi \in \mathcal{D}(\mathbf{R}), \quad (13)$$

where $\hat{\phi}(z) = \int_{-\infty}^{+\infty} \phi(s) e^{-2\pi i z s} ds$. Note that $[P_a, f]$ is different from $\langle P_a, \phi \rangle$. Notice also that the left-hand series in (13) is a *finite sum* for each fixed ϕ .

Note that, by virtue of Lemma 5, each C^∞ periodic function f satisfies

$$\forall n \geq 0, \exists a \text{ constant } C_n > 0 \text{ such that } |\hat{f}(k)| \cdot |k|^n \leq C_n, \forall k \in \mathbf{Z};$$

and furthermore, each periodic distribution S of order N satisfies

$$|\hat{S}(k)| = |[S, e^{-2\pi i k x/a}]| \leq C_N \|e^{-2\pi i k x/a}\|_{(N)} \leq C_N |k|^N, \quad \forall k \in \mathbf{Z}.$$

Conversely,

Theorem . If, $\forall n \geq 0$, $|c_k| \cdot |k|^n$ is bounded as $|k| \rightarrow \infty$, then the series $\sum c_k e^{2\pi i k x/a}$ is the F.S. of some $f \in \mathcal{P}_a$; and if $|c_k| \leq O(|k|^N)$ for some integer $N \geq 0$, then the series $\sum c_k e^{2\pi i k x/a}$ is the F.S. of some $S \in (\mathcal{P}_a)'$.

Next we explain how $\langle S, \phi \rangle$ can be defined for *every* periodic distribution $S \in (\mathcal{P}_a)'$ and *every* $\phi \in \mathcal{D}(\mathbf{R})$. That is, we show that $(\mathcal{P}_a)' \subset (\mathcal{D}')_a$. First define the mapping $\sigma_a : \mathcal{D}(\mathbf{R}) \rightarrow \mathcal{P}_a$ by the formula

$$(\sigma_a \phi)(x) := \sum_{k=-\infty}^{+\infty} \phi(x - ka). \quad (14)$$

Notice that this series is *more than convergent!* Indeed, for each fixed choice of $x \in \mathbf{R}$, it is a *finite* sum because ϕ has compact support. In fact, (14) is a finite sum whenever x is restricted to a bounded interval. Now we can *define*

$$\langle S, \phi \rangle := [S, \sigma_a \phi].$$

On the other hand, suppose that $T \in (\mathcal{D}')_a$. Then we can define $[T, f]$ for all $f \in \mathcal{P}_a(\mathbf{R})$ by the formula $[T, f] := \langle T, \varphi f \rangle$, where φ is *any* real-valued function in $\mathcal{D}(\mathbf{R})$ which satisfies $\sigma_a \varphi = 1$. Such functions φ are called C^∞ **unitary functions**, and the value of $\langle T, \varphi f \rangle$ is *independent* of which such unitary function φ is used. We now give one example of such a C^∞ unitary function in order to prove that they exist.

$$\varphi(x) := \frac{\int_{|x/a|}^1 e^{-1/s(1-s)} ds}{\int_0^1 e^{-1/s(1-s)} ds}, \quad \text{for } |x| \leq a, \quad \text{and } \varphi(x) := 0, \quad \text{for } |x| \geq a.$$

A very useful tool for investigating the pointwise convergence of trig series, is the **Abel Summation Formula**. Set $A_n = \sum_{k=1}^n a_k$. Then

Version #1:

$$\sum_{k=1}^n a_k b_k = A_n b_{n+1} + \sum_{k=1}^n A_k (b_k - b_{k+1}).$$

Version #2:

$$\sum_{k=p}^q a_k b_k = \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}) + A_q b_q - A_{p-1} b_p.$$

Try these on the series $\sum \frac{\sin(kx)}{k}$ and $\sum \frac{e^{2\pi i k x}}{k}$.