Invariants of Cubic Similarity

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1 Summary of the Known Problems & Reductions

We consider polynomial maps $F : \mathbf{C}^n \to \mathbf{C}^n$. The question, first raised by Keller [1] in 1939 for polynomials over the integers but now also raised for complex polynomials and, as such, known as **The Jacobian Conjecture** (**JC**), asks whether a *polynomial* map F with nonzero constant Jacobian determinant $\det F'(x)$ need be a **polyomorphism**: *injective* and also surjective with polynomial inverse. It suffices to prove *injectivity* because in 1960–62 it was proved, first in dimension 2 by Newman [2] and then in all dimensions by Białynicki-Birula and Rosenlicht [3], that, for polynomial maps, surjectivity follows from injectivity; and furthermore, under the hypothesis $\det F'(x) = \mathit{const} \neq 0$, the inverse $F^{-1}(x)$ will be polynomial, at least in the complex case, if the polynomial map is bijective. We call polynomial maps $F(x)$ satisfying $\det F'(x) = 1$ **Keller maps**. Thus we have the following three well-known and very difficult problems.

Problem#1: Classify all **Keller maps** $F : \mathbf{C}^n \to \mathbf{C}^n$. (Open for $n \geq 2$.)

Problem#2: Classify all **polyomorphisms** $F : \mathbf{C}^n \to \mathbf{C}^n$. (Open for $n \geq 3$.)

The van der Kulk-Jung Theorem: Every polyomorphism of the plane is a composition of a finite number of linear and triangular ones. This solves Problem#2 for $n = 2$.

For proofs of the van der Kulk-Jung Theorem see [4, 5] and [6].

Problem#3: Prove (or disprove) that every Keller map is a polyomorphism.
(Keller's Question **JC**. The converse is an easy exercise.)

In 1980 it was shown by Bass, Connell, and Wright [7], and also by Yagzhev [8], that to prove **JC** in every dimension $n \geq 2$ it suffices to prove it (but also in every dimension $n \geq 2$) for the special case of maps of the form $F(x) = x - H(x)$, where $H(x)$ is *homogeneous of degree three*: $H(tx) \equiv t^3 H(x)$. Nilpotence of the Jacobian $H'(x)$, $H'(x)^n \equiv 0$, is equivalent to Keller's condition that $F(x)$ have a nonzero constant Jacobian determinant; for a proof see [7] or [9, pages 112–113, Proof of Lemma 1(c)]. So $F(x)$ is a Keller map iff $H'(x)^n \equiv 0$.

Problem#4: Classify all cubic-homogeneous $H(x)$ with $H'(x)^n \equiv 0$. (Open for $n \geq 3$.)

Theorem: The Markus-Yamabe Conjecture $\Rightarrow$ Keller's Jacobian Conjecture.

Proof: If $F(x) = x - H(x)$ with $H(tx) \equiv t^3 H(x)$ and $H'(x)^n \equiv 0$, and if $F(x_1) = F(x_2)$ for some $x_1 \neq x_2$, then the system $\dot{x} = G(x) := -x + H(x) + x_1 - H(x_1)$ has both x_1 and x_2 as stationary points, even though $G'(x) = -I + H'(x)$ has all its eigenvalues equal to -1. This contradicts the Markus-Yamabe Conjecture. See [10, 11, 12, 13] and [14]. □

2 The Bilinear B-Matrix and Cubic-Similarity

Lemma 1. There is a unique bilinear mapping $\mathcal{B} : \mathbf{C}^n \times \mathbf{C}^n \to \mathcal{M}_n(\mathbf{C})$ such that

1. $\mathcal{B}(x, x) = H'(x)$. (Thus also $F(x) := x - \frac{1}{3}\mathcal{B}(x, x)x$.)
2. $\mathcal{B}(x, y) = \mathcal{B}(y, x)$.
3. $\mathcal{B}(x, y)z = \mathcal{B}(x, z)y$.

Proof: Namely, $\mathcal{B}(x, y) = H'\left(\frac{x+y}{2}\right) - H'\left(\frac{x-y}{2}\right)$. See [9] for further details.

Lemma 2. The **Mean-Value Formula** for cubic maps $F(x) = x - H(x)$:

$$F(x) - F(y) = \left[I - \frac{1}{3}\{\mathcal{B}(x, x) + \mathcal{B}(x, y) + \mathcal{B}(y, y)\}\right](x - y).$$

Corollary. The map $F(x) = x - H(x)$ is *injective* iff $\mathcal{B}(x, y)(x - y) = x - y$ implies $x = y$.

For a proof see [9, page 118, Section 3.3].

The B-Matrix Diagram

$$\begin{array}{ccc}
\mathcal{B}(x,y)^n \equiv 0 & \overset{\not\Longleftarrow}{\Longrightarrow} & \mathcal{B}(x,x)^n \equiv 0 \\
\not\Uparrow \ \Downarrow & \swarrow ? & \Uparrow \ (\Downarrow \ \textit{iff } \mathbf{JC}) \\
\begin{array}{c}\mathcal{B}(x,y)(x-y) = (x-y) \\ \textit{implies} \ \ x = y\end{array} & \Longleftrightarrow & \begin{array}{c}F(x) := x - \frac{1}{3}\mathcal{B}(x,x)x \\ \textit{is injective}\end{array}
\end{array}$$

Drużkowski's Reduction: It suffices to prove **JC** for maps of the special form

$$F_A(x) = x - H_A(x) :\equiv x - [diag(Ax)]^3\,\mathbf{1},$$

where A is an $n \times n$ complex matrix, $diag(x)$ denotes the diagonal matrix $diag[x_1, \ldots, x_n]$, and $\mathbf{1}$ denotes the column of n 1's. Drużkowski calls these **cubic-linear** maps.

For a proof see [15] or [16]. We call A the **kernel** of F_A and H_A.

For cubic-linear maps F_A we also have:

$$H'_A(x) = 3\,[diag(Ax)]^2\,A,$$

and

$$\mathcal{B}_A(x, y) = 3\,[diag(Ax)]\,[diag(Ay)]\,A.$$

Definition 1. A complex $n \times n$ matrix A is called **cubic-admissible** if $H'_A(x)$ is nilpotent for all $x \in \mathbf{C}^n$. That is, if the map $x \mapsto F_A(x)$ is a Keller map.

The change of vector variables $x = Pu$ and $y = Pv$ in the equation $y = F_A(x)$ leads to

Definition 2. Call matrices A and D **cubic-similar**, denoted $A \overset{cubic}{\sim} D$, if there exists an invertible complex matrix P such that, for all vectors $u \in \mathbf{C}^n$,

$$[diag(APu)]^3\,\mathbf{1} = P\,[diag(Du)]^3\,\mathbf{1}.$$

Three other conditions each equivalent to cubic-similarity:

(a) $[diag(APu)]^2 \, AP = P \, [diag(Du)]^2 \, D, \quad \forall u \in \mathbf{C}^n.$

(b) $P^{-1} H'_A(Pu) P = H'_D(u), \quad \forall u \in \mathbf{C}^n.$

(c) $P^{-1} \mathcal{B}_A(Pu, Pv) P = \mathcal{B}_D(u, v), \quad \forall u, \, v \in \mathbf{C}^n.$

Some Cubic-Similarity Invariants:

(1) *Nilpotence* of $H'_A(x)$.

(2) *Injectivity* of $F_A(x) := x - H_A(x)$. Matrix A is called **good** if F_A is injective.

(3) *Rank* of matrix A. (For nonzero admissible A, $1 \le rank(A) < n$.)

(4) *Nilpotence index* $\alpha(A)$ of $H'_A(x)$. ($1 \le \alpha(A) \le n$. A itself need not be nilpotent.)

(5) *Nilpotence* of $\mathcal{B}_A(x, y)$. (Matrix A is called **wonderful** if $\mathcal{B}_A(x, y)^n \equiv 0$.)

(6) *Nilpotence index* $\beta(A)$ of $\mathcal{B}_A(x, y)$. ($1 \le \beta(A) \le n$, or $\beta = \infty$ if $\mathcal{B}_A(x, y)^n \not\equiv 0$.)

(7) There are many other independent invariants whose significance is being investigated.

Representatives for Equivalence Classes:
Each 2×2 admissible matrix A can be written as a dyad

$$A = \begin{bmatrix} a \\ b \end{bmatrix} [\, -b^3 \quad a^3 \,] = \begin{bmatrix} -ab^3 & a^4 \\ -b^4 & ba^3 \end{bmatrix},$$

for some complex numbers a and b. Furthermore, every such A is wonderful: $\mathcal{B}_A(x, y)^n \equiv 0$. Each 2×2 nonzero admissible matrix is **cubic-similar** to the *one* representative

$$J(1.2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Each 3×3 nonzero admissible matrix is **cubic-similar** to one of the *two* representatives

$$J(1.2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad or \quad J(2.3) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The integers in their names denote *rank* of A and *nilpotence index* of $\mathcal{B}_A(x, x)$. See [17].

Next we list the six representatives from [17] for the cubic-similarity equivalence classes of the (4×4)-admissible matrices A .

$$J(1.2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad J(2.3) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad N(2.3) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J(2.2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad J(3.4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad N(3.4) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

They are admissible and mutually inequivalent with respect to cubic-similarity. According to a recent computer-check by Engelbert Hubbers, a masters student of Arno van den Essen in Nijmegen, The Netherlands, all 4×4 admissible matrices are cubic-similar to one of the *six* representatives listed above.

Below is a 15×15 matrix A which is good but not wonderful! It has rank 5, nilpotence index 2, and $\mathcal{B}_A(x,x)^5 = 0$. This example is a slight modification of the example given for another purpose on page 39 of [16]. It is easy to check that $\mathcal{B}_A(e_1, e_2)$ is not nilpotent (so A is not wonderful); and that $F(A)$ is injective (so that A is good). It is harder to show that $\mathcal{B}_A(x,x)$ has nilpotence-index 5 for all x in $\mathbf{C}^{15}$. This was checked by computer in a documented and repeatable manner [18, 19]. $A =$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 2 & -2 & -2 & -2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\
0 & 0 & 2 & 0 & 2 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & -2 & 4 & 0 & 0 & 0 & -2 & 2 & 2 & 2 & 0 & 0 & 0 & -2 \\
-2 & 0 & -2 & 4 & 0 & 0 & 0 & -2 & 2 & 2 & 2 & 0 & 0 & 0 & -2 \\
0 & -2 & -2 & 4 & 0 & 0 & 0 & -2 & 2 & 2 & 2 & 0 & 0 & 0 & -2 \\
-2 & 0 & 2 & 0 & 2 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\
-2 & 0 & 0 & 4 & 2 & -2 & -2 & -2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\
0 & -2 & 0 & 4 & 2 & -2 & -2 & -2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\
-2 & 0 & -2 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\
0 & -2 & 2 & -4 & 0 & 0 & 0 & 2 & -2 & -2 & -2 & 0 & 0 & 0 & 2 \\
0 & -2 & 0 & -4 & -2 & 2 & 2 & 2 & 0 & 0 & -2 & 0 & 0 & -2 & 0 \\
-2 & -2 & -2 & 4 & 0 & 0 & 0 & -2 & 2 & 2 & 2 & 0 & 0 & 0 & -2 \\
-2 & -2 & 0 & 4 & 2 & -2 & -2 & -2 & 0 & 0 & 2 & 0 & 0 & 2 & 0
\end{bmatrix}$$

The characteristic polynomial of $\mathcal{B}_A(x,y)$ is $\det[tI - \mathcal{B}_A(x,y)] = t^{15} + 576(x_1y_2 - x_2y_1)^2 t^{13}$. While this example and the representatives shown above are nilpotent, not every admissible matrix A need itself be nilpotent (see the 2×2 dyads above); and not every nilpotent is admissible. However, it follows from a result of Drużkowski [20] that every cubic-similarity equivalence class *contains* a nilpotent matrix. But this 15-dimensional example of a good-but-not-wonderful matrix shows that not every admissible matrix A is cubic-similar to a triangular matrix T: Because the matrix $\mathcal{B}_T(x,y)$ determined by a triangular matrix T is itself triangular, hence nilpotent for all x and y (so both T and A would be wonderful).

So far, we have found *eighteen inequivalent* simple representatives for cubic-similarity equivalence classes of 5×5 matrices; but there may be a few more. The hope is that we may discover some pattern that allows one to establish some general results for all dimensions n—so as not to rely forever on searches by computer.

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[19] G. H. Meisters. Inverting a Cubic-Linear Mapping in 15-Dimensions. A 6 page *Mathematica* Notebook. *Mathematica* Version 2.1 for NeXT Computers, Wolfram Research, Inc., July 1993. This inversion is also easy by hand!

[20] Ludwik M. Drużkowski. The Jacobian Conjecture in case of rank or corank less than three. *Journal of Pure and Applied Algebra* **85** (1993), 233–244. Proves (I): Every complex matrix A is cubic-similar to a complex matrix D with the properties (1) $D = H'_D(c)$, for some vector c, and (2) the nilpotence index of $H'_D(x) := 3[diag(Dx)]^2 D$ is the same as the nilpotence index of D itself. Also proves (II): If $rank(A)$ or $corank(A) := n - rank(A)$ is less than three, then $F_A := x - H_A(x)$ is tame; i.e., $F(x) - F(0)$ is a finite composition of linear automorphisms and nonlinear shears $T(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_i + f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), x_{i+1}, \ldots, x_n)$. In particular, every cubic-admissible matrix A is cubic-similar to a nilpotent matrix D.

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