Invariants of Cubic Similarity

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Trento Conference — September 14, 1993

Abstract

The question about polynomial maps $F : \mathbf{C}^n \to \mathbf{C}^n$, first raised by Keller [1] in 1939 for polynomials over the integers but now also raised for complex polynomials and, as such, known as **The Jacobian Conjecture** (**JC**), asks whether a *polynomial* map F with nonzero constant Jacobian determinant $\det F'(x)$ need be a **polyomorphism**: Injective and also surjective with polynomial inverse. The known reductions [2, 3, 4, 5] show that it suffices to prove *injectivity* for maps of Drużkowski's special **cubic-linear** form

$$F_A(x) = x - H_A(x) :\equiv x - [diag(Ax)]^3\,\mathbf{1},$$

where A is an $n \times n$ complex matrix, $diag(x)$ denotes the diagonal matrix $diag[x_1, \ldots, x_n]$, and $\mathbf{1}$ denotes the column of n 1's. The complex matrix A is the **kernel** of F_A and H_A. Then the Jacobian of the cubic-homogeneous part is $H'_A(x) = 3\,[diag(Ax)]^2\,A$ and the associated matrix-valued bilinear map is $\mathcal{B}_A(x, y) = 3\,[diag(Ax)]\,[diag(Ay)]\,A$. **Dfn 1**: A complex $n \times n$ matrix A is called **cubic-admissible** if $H'_A(x)$ is nilpotent $\forall x \in \mathbf{C}^n$; or, equivalently, $\det F'_A(x) \equiv 1$ (that is, $x \mapsto F_A(x)$ is a **Keller map**). See [6]. The change of vector variables $x = Pu$ and $y = Pv$ in the equation $y = F_A(x)$ leads to **Dfn 2**: Matrices A and D are **cubic-similar** ($A \overset{cubic}{\sim} D$) if, for an invertible matrix P, $[diag(APu)]^3\,\mathbf{1} = P\,[diag(Du)]^3\,\mathbf{1}$, $\forall u \in \mathbf{C}^n$; or $P^{-1}H'_A(Pu)P = H'_D(u)$, $\forall u \in \mathbf{C}^n$; or $P^{-1}\mathcal{B}_A(Pu, Pv)P = \mathcal{B}_D(u, v)$, $\forall u, v \in \mathbf{C}^n$. Some **Invariants of Cubic-Similarity** are: *Nilpotence* of $H'_A(x)$; the *nilpotence index* of $H'_A(x)$; *injectivity* of $F_A(x) := x - H_A(x)$; the *rank* of the matrix A; *nilpotence* of $\mathcal{B}_A(x, y)$; and the *nilpotence index* of $\mathcal{B}_A(x, y)$. There are many other invariants, independent of these, whose significance is being investigated. Each 2×2 admissible A can be written as a dyad $A = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} -b^3 & a^3 \end{bmatrix} = \begin{bmatrix} -ab^3 & a^4 \\ -b^4 & ba^3 \end{bmatrix}$, satisfies $\mathcal{B}_A(x, y)^n \equiv 0$, and is **cubic-similar** to the *one* representative $J(1.2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Each 3×3 admissible matrix A is **cubic-similar** to one of the *two* nilpotent Jordan representatives $J(1.2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $J(2.3) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The integers in their names denote *rank* of A and *nilpotence index* of $\mathcal{B}_A(x, x)$. According to a computer-check made (after the September 1993 Trento Conference) by Engelbert Hubbers, a masters student of Arno van den Essen in Nijmegen, The Netherlands, each 4×4 admissible matrix A is **cubic-similar** to one of the *six* mutually inequivalent representatives previously given in [6] as: $J(1.2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$J(2.3) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $N(2.3) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $J(2.2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $J(3.4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $N(3.4) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

We have found *eighteen inequivalent* representatives for cubic-similarity equivalence classes in 5-dimensions; there may be a few more. We hope to find some pattern that allows us to establish a general result for all dimensions n. There is a 15×15 matrix A [7] for which F_A is injective but $\mathcal{B}_A(x, y)^{15} \not\equiv 0$. It has rank 5, nilpotence index 2, $\mathcal{B}_A(x, x)^5 \equiv 0$, and the characteristic polynomial of $\mathcal{B}_A(x, y)$ is $t^{15} + 576(x_1y_2 - x_2y_1)^2 t^{13}$; see [8, 9]. Thus not every admissible matrix is cubic-similar to a triangular. Finally, every admissible is cubic-similar to a nilpotent matrix [10]. The bilinear matrix $\mathcal{B}_A(x, y)$ plays a fundamental and central role.

References

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