The TILF Story

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1 The TILF Question is Born

Early in 1969, during my seventh year at Colorado, my colleague Robert Richtmyer one day asked me (I still think quite innocently) whether I knew how to construct an example of a translation-invariant linear functional (a TILF) that is *not* continuous. (It had long been common knowledge that *continuous* TILFs are basically given by means of *integration*: $\phi \mapsto c \int_G \phi(t)\,dt$, for some constant c. At that time I had been intensively studying the Laurent Schwartz theory of *distributions*, which had been used with such dramatic effectiveness by Lars Hörmander to create the first completely general theory for solving linear partial differential equations (with constant coefficients). So, for Bob's question, I chose these; that is, linear functionals on the space $\mathcal{D}(\mathbb{R})$ of all infinitely differentiable functions of compact support, as the place to start looking for a *discontinuous* TILF! I felt confident that it would be merely an afternoon's work for me to produce an example of the kind that Bob Richtmyer wanted. But that afternoon came and went, as well as many more, before I began to realize that, in fact, I should rather be trying to prove that *all TILFs* on $\mathcal{D}(\mathbb{R})$ are **automatically** continuous!

2 First Results

But proving this was not easy—it took me several months before I finally saw how to do it. As I was excitedly writing-up the proof for publication, Peter Lax happened to come to Boulder to visit Stan Ulam for a few days. When I showed my astonishing result on the **automatic continuity of TILFs** on $\mathcal{D}(\mathbb{R})$ to Peter Lax, he exclaimed *"amazing and amusing!"* and he asked me to write-up a short research announcement for the *Bulletin of the American Mathematical Society* [1], to be published quicker than the full detailed proof to appear rather later in the *Journal of Functional Analysis* [2]. Shortly after this, sometime in 1970, while still at the University of Colorado-Boulder, I had the pleasure and good fortune of meeting Jean Dieudonné, the famous French mathematician who (along with four others) founded the celebrated "**Bourbaki**" group in the 1930's (the others were Henri Cartan, Claude Chevalley, Jean Delsarte, and André Weil). He seemed very interested in my result on the **automatic continuity of TILFs**; a few years later he sketched the main result of my papers [1,2] as the one-page outlined-problem #30, pages 207–208, and called it *Meisters' theorem*, in §18 of Chapter XXII of Vol. VI of his 1978 *Treatise on Analysis* [9].

3 TILFing Spreads

My Boulder colleague Wolfgang Schmidt, in his characteristic casual off-the-cuff manner, informed me one day of a clever technique that allowed me to complete the answer to the TILF question for the somewhat harder case of L^2 functions on the circle group $\mathbb{T}$; and this became the paper [3]. All this drove Richtmyer to get into the fray with [4]. Early on, I had discovered that TILFs *need not always be continuous*! Quickly after my discovery of **automatic continuity** for TILFs on $\mathcal{D}(\mathbb{R})$, I was able to easily construct examples of *discontinuous* TILFs on $L^1(\mathbb{R})$, as well as on $L^2(\mathbb{R})$ and a few other spaces [5]. The case of *discontinuous* TILFs was then pursued further by my student A. C. Serold [N59] and, most notably, by Gordon Woodward, among a growing list of others. And so, contrary to what a few eminent mathematicians at first thought, the answer to the TILF question is neither "discontinuous TILFs exist on any space" nor "TILFs are automatically continuous on every space",—it really all depends (in some mysterious way) on the choice of the group G and the vector space $\mathcal{F}(G)$! And, in every case, some sort of Diophantine inequalities seem to play a crucial role! In the 70's I began receiving mail from many people around the world inquiring about TILFs. By far the most persistent of these correspondents was Lilian Asam of München, Germany. Finally, my wife and I met Lilian (and her charming husband, Peter) when we travelled through Germany in July of 1983 on our way to the 1982-WARSZAWA ICM, which was held in August of 1983. Lilian became a devoted TILFer; both her masters' and doctors' theses dealt with TILFs,—and she later produced the nice paper [N2]. Lilian Louise Colombe Asam (née Graue), born 18/July/1953, was killed in a tragic boating accident on the Starnbergersee near München on September 12, 1987; she had returned the corrected page-proofs for her paper [N2], but it appeared after her death.

My papers [1,2] were thus the first of a series of results, now called **automatic continuity of translation-invariant linear forms**. They inspired many others, including Lilian Asam [N2], Jean Bourgain [N8], Alain Connes [see N32], Barry E. Johnson [N28], C. J. Lester [N32], Peter Ludvik [N36], Rodney Nillsen [12], L. Nylund [N48], Walter Roelcke (Lilian Asam's thesis advisor in München), Joseph Rosenblatt [N51], Sadahiro Saeki [N55], Arno C. Serold [N59], Y. Takahashi [N67], George A. Willis [N70], and Gordon Woodward [N71], to enter this area with theorems of their own: See Nillsen's book [12] for all these references here denoted [N #].

4 Nillsen's Book

In 1976, Dieudonné invited me to write a monograph on TILFs in a series he was editing. At the time I received that invitation I was leaving the area of TILFs (although I continue to this day to referee TILF papers and to write reviews of many TILF papers for the journal **Math Reviews**)—I was getting deeply involved in another area (polynomial mappings) and so I put Dieudonné off, saying neither "yes" nor "no"; but I never found the time to do it. Now I am very glad that I didn't try to do it, because Rodney Nillsen has done it with his book [12] (and he did a far better job than I could have done). Nillsen's book fulfills the request that Dieudonné made of me in 1976 to write a monograph on TILFs; Jean Alexandre Dieudonné [1906–1992] would have liked Nillsen's book very, very much—of this I am quite certain! And my conscience is free of the obligation I had felt heavily from 1976 to 1994!

5 Remaining Questions

There remain many interesting questions about TILFs. For example, no one ever developed a general answer to the question: Which LCTV-spaces $\mathcal{F}(G)$ of distributions over which groups G exhibit this **automatic continuity of TILFs**, and which ones do not? Perhaps this question will never be answered in full, but many strange surprises await those who seek an answer! In the meantime, Nillsen's book [12] will serve for a long time to come as the best source of information in this area. It is completely accessible to the beginner, as well as being indispensable to the most advanced researcher. Furthermore, Nillsen has connected the TILF question with other parts of the wider area of harmonic analysis and, in particular, with the exciting new area of **wavelets**.

(The notation [N #] used above refers to the references in Nillsen's book [12].)

The TILF Story References Chronologically Ordered

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