Polynomial Maps of n-Space

Injectivity, Surjectivity, Inverse

The 1939 Jacobian Conjecture

of Ott-Heinrich Keller

The Markus-Yamabe Conjecture

R. I. P.

MYC 1960–1995

Many Examples & Open Questions

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NOTATION

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ is a column vector in } \mathbb{R}^n \text{ or } \mathbb{C}^n$$

$F'(x)$ denotes the **Jacobian Matrix** (derivative) of the $\mathcal{C}^1$ map $F : \mathbb{R}^n \to \mathbb{R}^n$

$$\left[F'(x)\right]_{ij} = \partial F_i / \partial x_j$$

$$\text{diag}(x) := \begin{pmatrix} x_1 & & & & & \\ & x_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & x_n \end{pmatrix}$$

$$\text{Define } \mathbf{1} := \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \text{ so that } \text{diag}(x)\mathbf{1} = x.$$

Keller's Jacobian Conjecture (1939)

***** Ott-Heinrich Keller [1906–1990] *****

$F : \mathbb{C}^n \to \mathbb{C}^n$ polynomial & $\det F'(x) \equiv 1$
$\Rightarrow F$ is bijective with polynomial inverse.

<u>CONVERSE</u>: F **polyomorphism** $\Rightarrow \det F'(x) \equiv 1$.
<u>PROOF</u>: $I = x' = [G(F(x))]' = G'(F(x))F'(x)$.
So $\det I = 1 = \det G'(F(x)) \det F'(x)$. □

REDUCTIONS: It suffices to prove

1. **injectivity** of F

   [Białynicki-Birula & Rosenlicht (1962); and Rudin (1995)]

2. for **cubic-homogeneous** maps
   $F(x) := x - H(x)$ with $H(tx) = t^3 H(x)$
   [Yagzhev; Bass, Connell & Wright (c. 1980)]

3. or merely for **cubic-linear** maps
   $F_A(x) := x - H_A(x)$ where
   $H_A(x) := [\mathrm{diag}(Ax)]^3 \mathbf{1} = [\mathrm{diag}(Ax)]^2 Ax$,
   for some $n \times n$ matrix A. [Drużkowski (1983)]

Dfn: Call matrix A **admissible** if $\det F'_A(x) \equiv 1$.

Dfn: Call matrix A **good** if F_A is injective.

By converse of JC, *every good matrix is admissible*.

JC is converse: *Is every admissible matrix good*?

We start with an older question.

What condition(s) on the Jacobian derivative $F'(x)$ of a $\mathcal{C}^1$ map $F : \mathbb{R}^n \to \mathbb{R}^n$ force F to be injective?

Some old answers for $\mathcal{C}^1$ maps $f : \mathbb{R}^n \to \mathbb{R}^n$.

1. **The Mean-Value Theorem on $\mathbb{R}^1$** $\Rightarrow$
   f is injective if $f'(x) \neq 0$ on $\mathbb{R}^1$. Proof:
   $f(x) - f(y) = f'(\theta)(x - y)$ & $f(x) = f(y) \Rightarrow x = y$. $\square$
2. **The Local Inverse Function Theorem:**
   $\det F'(z) \neq 0 \Rightarrow$ *F is injective near z.*
   In fact, F is a diffeomorphism of an open neighborhood of z onto an open neighborhood of $F(z)$.
3. **Hadamard's Global Inverse Func. Thm:**
   A $\mathcal{C}^1$ map $F : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism iff f is proper and $\det F'(x) \neq 0$ *on $\mathbb{R}^n$.*

Dfn. A continuous map $F : \mathbb{R}^n \to \mathbb{R}^n$ is called **proper** if it satisfies one of the equivalent conditions:

(a) K compact $\Rightarrow F^{-1}(K)$ compact.

(b) $\|x_k\| \to \infty \Rightarrow \|F(x_k)\| \to \infty$.

(c) F is **closed** & $F^{-1}(y)$ is compact $\forall y \in \mathbb{R}^n$.

A curious thing about nonconstant polynomial maps of one complex variable $w = f(z)$:

1. They are open maps.
2. They are proper maps.
3. They are closed maps.

THEOREM: (Meisters/Olech/van den Essen)
If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map with $\det T' \neq 0$, then T closed $\Rightarrow$ T proper $\Rightarrow$ T injective.

The Jacobian Conjecture is False for Analytic Maps:

Define the mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$F : \begin{cases} u(x,y) &= \sqrt{2}\exp(x/2)\cos(y\exp(-x)) \\ v(x,y) &= \sqrt{2}\exp(x/2)\sin(y\exp(-x)) \end{cases}$$

Then $\det F'(x) \equiv 1$, but F is not injective:

$$F(0, y + 2k\pi) = F(0, y).$$

The Jacobian Conjecture is False for Rational Maps (Vitushkin)

Define the mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$F : \begin{cases} u(x,y) \; = \; x^2y^6 + 2xy^2 \\ v(x,y) \; = \; xy^3 + 1/y \end{cases}$$

Then $\det F'(x,y) \equiv -2$, but F is not injective:

$$F(-3,-1) = F(1,1) = (3,2).$$

Vitushkin's Rational Map Generalized

Define the mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$F : \begin{cases} u(x,y) \; = \; ax^2y^6 - (\delta/d)xy^2 \\ v(x,y) \; = \; -2a(d^2/\delta)xy^3 + dy^{-1} \end{cases}$$

Then $\det F'(x,y) \equiv \delta$, but F is not injective:

Find values for (x_1, y_1), (x_2, y_2), and (u_0, v_0) so that

$$F(x_1,y_1) = F(x_2,y_2) = (u_0,v_0).$$

Are the eigenvalues of $\det F'(x)$ bounded from zero?

The Jacobian Conjecture is False for Nice Rational Maps (Meisters)

Define the mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$F : \begin{cases} u(x,y) & = & x - \frac{p}{p+1}(\frac{x}{y})^{p+1} \\ v(x,y) & = & y - (\frac{x}{y})^p. \end{cases}$$

Then $\det F'(x) \equiv 1$, $\operatorname{trace} F'(x) = 2$, and eigenvalues $\{1, 1\}$ are bounded from zero,—

but F is **not injective**:

For example, when $p = 1$,

$$F(6,3) = F(12,-3) = (4,1).$$

The Jacobian Conjecture is True for All Known Polynomial Maps. Poly Ex # 1: (Hénon's Map & Chaos)

Define the polynomial mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$F : \begin{cases} u(x,y) &= y + 1 - ax^2 \\ v(x,y) &= bx. \end{cases}$$

Then $\det F'(x) \equiv -b$, $\operatorname{trace} F'(x) = -2ax$, and the eigenvalues of $F'(x)$ are

$$\{ -ax - \sqrt{b + a^2x^2}, -ax + \sqrt{b + a^2x^2} \}.$$

F is injective with inverse

$$F^{-1} : \begin{cases} x &= v/b \\ y &= -1 + u + av^2/b^2. \end{cases}$$

A good choice for **chaos** under iteration is

$$(a,b) = (1.4, 0.3).$$

Friedland & Milnor,
Dynamical properties of plane polynomial automorphisms,
Ergodic Theory & Dynamical Systems, **9** (1989) 67–99.

Each conjugacy class in $\mathbf{GA}_2(\mathbb{C})$ contains either an **affine** map, a **triangular** map, $T(x,y) := (\alpha x + p(y), \beta y + \gamma)$ with $\alpha\beta \neq 0$, or a composition of **generalized Hénon** maps

$$H(x,y) := (y, p(y) - \delta x), \quad \deg p \geq 2, \quad \delta \neq 0.$$

A 1–1 Analytic Map with Eigenvalues Bounded from Zero (Meisters)

Define the mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$F : \begin{cases} u(x, y) &= 2x + \cos(x)\cos(y) \\ v(x, y) &= 2y - \cos(x)\cos(y). \end{cases}$$

Then

$$\det F'(x) = 4 + 2\sin(y - x),$$
$$\operatorname{trace} F'(x) = 4 + \sin(y - x),$$

and the eigenvalues of $F'(x)$,

$$\{2, 2 + \sin(y - x)\}$$

are bounded away from zero.

This F is injective:

Can you prove this? Can you find its inverse?

Can you solve explicitly for (x, y) in terms of (u, v)?

The Strong Real Jacobian Conj is False!

$F : \mathbb{R}^n \to \mathbb{R}^n$ poly & $\det[F'(x)] \neq 0$ on $\mathbb{R}^n$

$\Rightarrow F$ is bijective with analytic inverse.

Example: The map $x \mapsto x + x^3 \in \mathbb{R}^1$ has Jacobian $1 + 3x^2 \geq 1$ and is bijective with analytic inverse.

Counterexample (Serguey Pinchuk May 1994):

Define the polynomial map $\langle x, y\rangle \mapsto \langle p, q\rangle$ as follows:

$$\begin{aligned}
t &= xy - 1, \\
h &= t(xt + 1), \\
f &= ((h+1)/x)(xt+1)^2, \\
p(x,y) &= f + h, \\
Q &= -t^2 - 6th(h+1), \\
u &= 170fh + 91h^2 + 195fh^2 + 69h^3 \\
&\quad +75h^3 f + (75/4)h^4, \\
q(x,y) &= Q - u.
\end{aligned}$$

Then the Jacobian J is a sum of squares and $\neq 0$:

$$J \equiv \partial(p,q)/\partial(x,y) = t^2 + (t + (13 + 15h)f)^2 + f^2.$$

$J = 0 \Rightarrow t = 0$ & $f = 0$; then $h = 0$ & $f = 1/x \neq 0$.

Yet, the map is not injective because

$$\langle p, q\rangle(1,0) = \langle p, q\rangle(-1,-2) = \langle 0, -1\rangle$$

Q.E.D.

Pinchuk's Counterexample to SRJC Written Out Explicitly

Degree$(p, q) = (10, 25)$

$$
\begin{aligned}
p(x,y) \;=\; & (1 - x + x^2y) \\
& (-x + y - 2xy + 3x^2y + 2x^2y^2 \\
& -3x^3y^2 + x^4y^3)
\end{aligned}
$$

$$
\begin{aligned}
q(x,y) \;=\; & ((-1 + xy)(-33 + 244x - 398x^2 \\
& +536x^3 - 645x^4 + 300x^5 \\
& -200y + 623xy - 1424x^2y \\
& +3386x^3y - 5980x^4y + 6315x^5y \\
& -2700x^6y - 180xy^2 - 515x^2y^2 \\
& +4516x^3y^2 - 14520x^4y^2 + 26360x^5y^2 \\
& -26145x^6y^2 + 10800x^7y^2 - 300x^2y^3 \\
& +1725x^3y^3 - 10236x^4y^3 + 33224x^5y^3 \\
& -60860x^6y^3 + 60375x^7y^3 - 25200x^8y^3 \\
& -1800x^4y^4 + 11400x^5y^4 - 40442x^6y^4 \\
& +80680x^7y^4 - 85575x^8y^4 + 37800x^9y^4 \\
& -4500x^6y^5 + 24750x^7y^5 - 62036x^8y^5 \\
& +76545x^9y^5 - 37800x^{10}y^5 - 6000x^8y^6 \\
& +25800x^9y^6 - 42315x^{10}y^6 + 25200x^{11}y^6 \\
& -4500x^{10}y^7 + 13245x^{11}y^7 - 10800x^{12}y^7 \\
& -1800x^{12}y^8 + 2700x^{13}y^8 - 300x^{14}y^9))/4.
\end{aligned}
$$

JC is True for Quadratic Maps in all Dimensions

Quadratic maps $F : \mathbb{C}^n \to \mathbb{C}^n$ satisfy

The Mean-Value Formula

$$F(x) - F(y) = F'\left(\frac{x+y}{2}\right)(x-y).$$

Consequently,

$$\det F'(x) \equiv \text{constant} \neq 0$$

implies that $F(x)$ is *injective*.

Let's begin a list of Open Problems

This page, especially Remark 1, was revised January 21, 1997.

Open Q1. $\mathbf{JC}(\mathbb{F}, n)$: For any field $\mathbb{F}$ of characteristic zero, any integer $n \geq 2$, and any polynomial mapping $P : \mathbb{F}^n \to \mathbb{F}^n$, the condition $\det P'(x) \in \mathbb{F} \setminus \{0\}$ (also written $\det P'(x) \equiv c \neq 0$) implies that P is *injective* and *surjective*, and P^{-1} is *polynomial.*

Remark 1: If $\mathbb{F}$ is a field of prime characteristic p, the map $f : \mathbb{F} \to \mathbb{F}$ defined by $f(x) := x + x^p$ has *no polynomial inverse* even though $\det f'(x) \equiv 1$. For if $f(g(y)) = y$ with $\deg g = d$, then $1 < pd = \deg f(g(y)) = \deg y = 1$. A contradiction!

Remark 2: For $\mathbf{JC}(\mathbb{C}, n)$ it suffices to prove P injective: *Injective polynomial maps of* $\mathbb{C}^n \hookleftarrow$ *are polyomorphisms.* Also, *Injective polynomial maps of* $\mathbb{R}^n \hookleftarrow$ *are surjective.*

[Newman, *Proc. A. M. S.* **11** (1960) 867–870 ($n = 2$); Białynicki-Birula & Rosenlicht, *Proc. A. M. S.* **13** (1962) 200–203 ($n \geq 2$); J. Ax, in D. J. Lewis, *Proc. Symp. Pure Mathematics XX* (1969) 161–163; Rudin, *Amer. Math. Monthly* **102** (No. 6) June-July (1995) 540–543, a simple treatment.]

Remark 3: $\mathbf{JC}(\mathbb{R}, n)$ asks if $\det P'(x) \equiv c \neq 0$ implies *injectivity* for real polynomial maps $P : \mathbb{R}^n \to \mathbb{R}^n$. And if so, must the inverse P^{-1} be polynomial ?

Open Q2. Is there a *non-injective* $\mathcal{C}^1$-map $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $\det F'(x) \equiv 1$ and all the eigenvalues of $F'(x)$ are bounded away from zero for all $x \in \mathbb{R}^n$?

Open Q3. Is there a *non-injective* polynomial map $P : \mathbb{R}^n \hookleftarrow$ such that all eigenvalues of $F'(x)$ are bounded away from zero for all $x \in \mathbb{R}^n$? Pinchuk's example?

Another 1–1 Analytic Map with Eigenvalues Bounded from Zero (Meisters)

Define the mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$F : \begin{cases} u(x, y) \;=\; x - \exp(\cos^2 y) \\ v(x, y) \;=\; 2y + \exp(\cos^2 y). \end{cases}$$

Then

$$\det F'(x) = 2 - (\sin 2y) \exp(\cos^2 y),$$

$$\operatorname{trace} F'(x) = 3 - (\sin 2y) \exp(\cos^2 y),$$

and the eigenvalues of $F'(x)$,

$$\{\, 1, 2 - (\sin 2y) \exp(\cos^2 y) \,\},$$

are bounded away from zero.

This F is injective by Hadamard's Theorem.

Can you prove this? Can you find its inverse?

Can you solve explicitly for (x, y) in terms of (u, v)?

Structure of the groups $\mathbf{GA}_2(\mathbb{R})$ and $\mathbf{GA}_2(\mathbb{C})$

Dfn. A **polyomorphism** ψ of $\mathbb{F}^2$ is a bijective map such that both ψ and ψ^{-1} have polynomial components. $\mathbf{GA}_2(\mathbb{F})$ denotes the group of all such polyomorphisms ψ of $\mathbb{F}^2$.

Dfn. A **triangular** polyomorphism of $\mathbb{F}^2$ is defined by

$$T(x,y) := (\alpha x + p(y), \beta y + \gamma) \text{ with } \alpha\beta \neq 0 \text{ and } p \in \mathbb{F}[y].$$

Thm of Jung [1942]. For any field $\mathbb{F}$, $\mathbf{GA}_2(\mathbb{F})$ is generated by its **affine** and **triangular** subgroups $\mathbf{Af}_2(\mathbb{F})$ and $\mathbf{Tr}_2(\mathbb{F})$.

In fact, $\mathbf{GA}_2(\mathbb{F})$ is the **free product** of these two subgroups **amalgamated** over their intersection Δ. This is written

$$\mathbf{GA}_2(\mathbb{F}) = \mathbf{Af}_2(\mathbb{F}) *_\Delta \mathbf{Tr}_2(\mathbb{F}),$$

and means that each $\psi \in \mathbf{GA}_2(\mathbb{F})$ can be uniquely written in the form $\psi = A \circ T_1 \circ T_2 \circ \cdots \circ T_m$, where $A \in \mathbf{Af}_2(\mathbb{F})$, and the T_k, for $1 \leq k \leq m$, alternate between representatives of the nontrivial Δ-cosets of $\mathbf{Af}_2(\mathbb{F})$ and $\mathbf{Tr}_2(\mathbb{F})$. Equivalently, and more explicitly, each $\psi \in \mathbf{GA}_2(\mathbb{F})$ can be written in the form $\psi = L \circ S_1 \circ S_2 \circ \cdots \circ S_n$, where $L \in \mathbf{GL_2}(\mathbb{F})$, and each $S_i = S(p_i, \vec{v}_i)$ is the **nonlinear shift map** of $\mathbb{F}^2$ into itself, defined, for each $\vec{x} = (x,y)^*$, by $S(p,\vec{v})\vec{x} := \vec{x} + p(\vec{x}^* \vec{v}^\perp)\vec{v}$, where $p(t) \in \mathbb{F}[t]$ and $\vec{v}$ is a vector in $\mathbb{F}^2$. Here $\vec{x}^*$ denotes the transpose of the (column) vector $\vec{x}$, and $\vec{v}^\perp := (-y,x)^*$ is called "v-perp".

See Meisters, *Rocky Mountain J. Math.* **12** (1982) 679–705. See also van der Kulk [1953]; and S. Friedland & J. Milnor, Dynamical properties of plane polynomial automorphisms, *Ergodic Theory & Dynamical Systems*, **9** (1989) 67–99.

Each conjugacy class in $\mathbf{GA}_2(\mathbb{R})$ contains either an **affine** map, a **triangular** map, $T(x,y) := (\alpha x + p(y), \beta y + \gamma)$ with $\alpha\beta \neq 0$, or a composition of **generalized Hénon** maps

$$H(x,y) := (y, p(y) - \delta x), \quad \deg p \geq 2, \quad \delta \neq 0.$$