

Power-Similarity: Summary of First Results

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1 Brief History & Motivation

Around 1960, Markus & Yamabe [1] conjectured that, for each $n \geq 2$, every rest point x_0 of a nonlinear, class \mathcal{C}^1 , n -dimensional system of differential equations $\dot{x} = F(x)$ is *globally* asymptotically stable if the eigenvalues of the jacobian matrix $F'(x)$ have negative real parts at every point x in \mathbf{R}^n . At that time, Olech [2] connected this problem with the question of *injectivity* of the mapping $x \mapsto F(x)$. An older question, evidently first raised by Keller [3] in 1939 for polynomials over the integers but now also raised for complex polynomials, and, as such, known as “The Jacobian Conjecture” (**JC**), asks whether *polynomial* maps $F(x)$ with nonzero constant jacobian determinant $\det F'(x)$ need be *injective* (and also surjective with polynomial inverse). It suffices to prove *injectivity* because in 1960–62 it was proved, first in dimension 2 by Newman [4] and then in all dimensions by Białynicki-Birula and Rosenlicht [5], that, for polynomial maps, surjectivity follows from injectivity; and furthermore, under the hypothesis $\det F'(x) = \text{const} \neq 0$, the inverse $F^{-1}(x)$ will be polynomial, at least in the complex case, if the polynomial map is bijective.

About 1980 it was shown by Bass, Connell, and Wright [6], and also by Yagzhev [7], that to prove **JC** in every dimension $n \geq 2$ it suffices to prove it (but also in every dimension $n \geq 2$) for the special case of maps of the form $F(x) = x - H(x)$, where $H(x)$ is *homogeneous of degree three*. Using this “Reduction of Degree” result, it is easy to show that *the truth of the Markus-Yamabe Conjecture implies the truth of JC*: For suppose that, in some dimension n , there is a mapping $F(x) := x - H(x)$ that is not injective, where $H(tx) \equiv t^3 H(x)$ and $H'(x)^n \equiv 0$. (This nilpotence of $H'(x)$ is equivalent to Keller's condition that $F(x)$ have a nonzero constant jacobian determinant. For a proof, see e.g., [8, Lemma 1(c)].) Then $F(x_1) = F(x_2)$ for some $x_1 \neq x_2$ in \mathbf{C}^n . So, by the Markus-Yamabe Conjecture, both x_1 and x_2 are globally asymptotic rest points of the system

$$\frac{dx}{dt} = G(x) := -x + H(x) + x_1 - H(x_1),$$

because $H'(x)^n \equiv 0$ implies that the jacobian $G'(x) \equiv -I + H'(x)$ has all its eigenvalues equal to -1. But two distinct points cannot *both* be globally asymptotically stable. This contradiction proves that $F(x)$ must be injective after all. \mathcal{QED} \square

2 Reduction to Certain Complex Matrices

2.1 Cubic-Linear Maps

Thus it becomes interesting to try to classify the maps of the form $x \mapsto H(x)$, where $H(x)$ is homogeneous of degree three and $H'(x)$ is nilpotent for every x . This seems to be an extremely complicated and difficult problem. But in 1983, Drużkowski [9] gave a further reduction of the Jacobian Conjecture: It suffices to consider (but again in every dimension $n \geq 2$) maps of the even more special “cubic-linear” form $F_A(x) = x - (Ax)^3$, where A is an $n \times n$ complex matrix, and $(x^3)_i = (x_i)^3$, $1 \leq i \leq n$. The jacobian matrix of the mapping $x \mapsto (Ax)^3$ can be expressed as $3[\text{diag}(Ax)]^2 A$, where $x \mapsto \text{diag}(x)$ is the isomorphism of column vectors onto diagonal matrices. Using this notation one can write $(Ax)^k = [\text{diag}(Ax)]^k \mathbf{1}$, where $\mathbf{1}$ denotes the column vector of all 1's. All three notations $H_A(x) = (Ax)^k = [\text{diag}(Ax)]^k \mathbf{1}$, for $k \geq 1$, are useful. We need only consider matrices A for which $[\text{diag}(Ax)]^2 A$ is nilpotent $\forall x$. **What shall we name these matrices A ?**

2.2 Polyomorphisms of k^n vs Automorphisms of $k[x_1, \dots, x_n]$

Let k denote any field. A polynomial map F of k^n is a (usually nonlinear and non-multiplicative) mapping of the n -dimensional vector space k^n into itself such that each component of F is a polynomial in the n variables x_1, \dots, x_n ; i.e., $F = (F_1, \dots, F_n)$ and each F_j belongs to the polynomial ring $k[x] = k[x_1, \dots, x_n]$. On the other hand, a k -algebra homomorphism f of the polynomial ring $k[x]$ is a multiplicative and linear mapping of $k[x]$ into itself. The elements of k^n are column vectors $c = (c_1, \dots, c_n)^*$ of field elements c_j , while the elements of $k[x]$ are polynomials $p(x)$ in the n variables x_1, \dots, x_n . The following notations are in use:

- $\mathbf{GA}(k^n)$ = the group of all polynomial bijections of k^n with polynomial inverses.
- $\text{Aut } k[x]$ = the group of all automorphisms f of $k[x_1, \dots, x_n]$.

Both $\mathbf{GA}(k^n)$ and $\text{Aut } k[x]$ form groups with respect to composition of maps. These two groups enjoy a natural anti-isomorphism defined by

$$f \mapsto F \text{ where } F_j = f(x_j), \quad \forall f \in \text{Aut } k[x], \text{ and } \forall F \in \mathbf{GA}(k^n).$$

This works because each homomorphism f is uniquely determined by its values at the variables x_1, \dots, x_n by virtue of its linearity and its multiplicativeness. In order to distinguish the elements of the group $\mathbf{GA}(k^n)$ from those of the group $\text{Aut } k[x]$, I use the name **polyomorphism** for elements of the former, and the name **automorphism** for the elements of the latter. Automorphisms f satisfy

$$f(p + q) = f(p) + f(q) \text{ and } f(pq) = f(p)f(q),$$

while polyomorphisms do not; furthermore, polyomorphisms and automorphisms have distinct domains and ranges. Thus, although the two groups $\mathbf{GA}(k^n)$ and $\text{Aut } k[x]$ are anti-isomorphic, they are not identical; it seems useful to maintain the distinction. Indeed, the ultimate resolution of **JC** may very well require both points of view, the one of analysis and the other of algebra, passing back and forth between them as necessary. The Jacobian Conjecture **JC** has its roots deep in both camps! I here adopt the first point of view; but it is wise to keep these ideas in mind for longer-term goals—and also, of course, in the case of unpredictable serendipitous discoveries.

3 The Equivalence Relation “Power-Similarity”

Thus the problem arises to classify, for each $n \geq 2$, all $n \times n$ complex matrices A for which the jacobian matrix $H'_A(x) \equiv 3[\text{diag}(Ax)]^2 A$ is nilpotent for all vectors $x \in \mathbf{C}^n$. Such matrices A will be called **admissible** matrices for lack of a better name; and the smallest integer $\alpha = \alpha(A) \geq 2$ for which $([\text{diag}(Ax)]^2 A)^\alpha = 0 \ \forall x$ will be called the **admissibility index** of A . Clearly, $2 \leq \alpha(A) \leq n$. I began working on this problem in the spring of 1992 and spoke about it on October 14 at the Conference on Polynomial Automorphisms organized by Arno van den Essen and held at C. I. R. M. Luminy, France, October 12–16.

Definition 1: For each $k \geq 1$, an $n \times n$ matrix A is called **k -ADMISIBLE** iff $([\text{diag}(Ax)]^{k-1} A)$ is nilpotent $\forall x$. In this general case, $H_A(x) = (Ax)^k$ so that $H'_A(x) = k([\text{diag}(Ax)]^{k-1} A)$. The integer k is called the “**degree of admissibility**” of A . Perhaps “ **k -nil matrices**” would be a better name than “ **k -admissible matrices**”.

Remark 1: *1-admissible* (or *1-nil*) is the same as *nilpotent*.

Remark 2: In addition to the *rank* of A there are the following four integer parameters:
 $n =$ **Dimension** (number of variables x_1, x_2, \dots, x_n).

The size of A is $n \times n$. All values of $n \geq 2$ must be considered.

$\nu = \nu(A) =$ The **nilpotence index** of A , when A is nilpotent.

$k =$ The **degree of admissibility** of the matrix A .

We are primarily interested in the cases $1 \leq k \leq 3$.

The case of primary interest is $k = 3$. A matrix can be k -admissible for more than one value of k . E. g., $N(2.3)$ of Prop. 12 is both 2- and 3-admissible.

$\alpha_k = \alpha_k(A) =$ The **admissibility index** of A corresponding to degree k .

The k is dropped when its value does not change in a given context.

Remark 3: The admissibility index of A is the nilpotence index of $H'_A(x)$.

Remark 4: $\text{Det } H'_A(x) = 0 \ \forall x$ iff $\text{det } A = 0$. Hence every admissible matrix is singular.

Remark 5: Not every 3-admissible matrix is nilpotent; nor is every nilpotent matrix necessarily 3-admissible.

Problem 1:

Classify all k -admissible $n \times n$ matrices (over \mathbf{C}) for each dimension $n \geq 2$.

The case $k = 3$ is of primary interest, although the case $k = 2$ should, perhaps, be considered at the same time (or even first).

The following equivalence relation seems to be fundamental for this problem.

Definition 2: Two $n \times n$ matrices A and B are **POWER- k -SIMILAR** iff there is a nonsingular matrix P , such that $(\text{diag}[APu])^k \mathbf{1} = P(\text{diag}[Bu])^k \mathbf{1} \ \forall u \in \mathbf{C}^n$. In the H notation (at the bottom of page 1) this is $H_A(Pu) \equiv PH_B(u)$. It corresponds to the change of variables, $x = Pu$ and $y = Pv$, in the equation $y = x - H_A(x) = x - (Ax)^k$.

I use the notation $A \text{ PkS } B$, or $A \tilde{k} B$, for this relation between A and B .

Problem 2:

Prove that $F_A(x)$ ($= x - (Ax)^3$) is injective whenever A is 3-admissible.

This is, of course, a reformulation of **JC**. I do not expect to settle this question, but I believe in Serendipity; so it serves as a beacon (or Holy Grail).

4 Summary of First Results

Here are a few simple but important facts:

Proposition 1. For each $k \geq 1$ the relation $A PkS B$ is an EQUIVALENCE RELATION. When $k = 1$, it is ordinary similarity of matrices. Many results stated only for $P3S$ hold just as well for all PkS , but I am primarily concerned with $P3S$. The PkS -equivalence classes are different for different k .

Proposition 2. If A is 3-admissible and $A \tilde{\cong} B$, then B is also 3-admissible.

Proposition 3. If F_A is injective and $A \tilde{\cong} B$, then F_B is also injective.

Corollary. In order to verify **JC** ($\forall \dim n \geq 2$) it suffices to check **JC** ($\forall \dim n \geq 2$) for only *one* matrix A in each $P3S$ -equivalence class.

Proposition 4. If $A \tilde{\cong} B$ and $Ae = \lambda e$ (that is, λ is an eigenvalue of A) and $u = P^{-1}e$, where P satisfies $(diag[APx])^3 \mathbf{1} = P(diag[Bx])^3 \mathbf{1} \quad \forall x$, then

$$(Bu)^3 = [diag(Bu)]^3 \mathbf{1} = \lambda^3 P^{-1} e^3, \text{ or } Bu = \lambda (P^{-1} e^3)^{1/3}.$$

Thus eigenvalues are not always preserved by the equivalence relation $P3S$, but nevertheless a useful formula is obtained.

Corollary. If $A \tilde{\cong} B$, then $Rank(A) = Rank(B)$.

Proposition 5. If A is 3-admissible, then there is a matrix D such that

1. $A \tilde{\cong} D$, and
2. $D = H'_D(c)$ for some vector $c \in \mathbf{C}^n$. (Say D has **Property \mathcal{D}** .)
3. Therefore A and D have the same admissibility index.

Proof: Part (3) follows from (1) because differentiation of $H_A(Pu) = PH_D(u)$ with respect to u yields the similarity relation $H'_A(Pu) = PH'_D(u)P^{-1}$ between the two jacobians $H'_A(Pu)$ and $H'_D(u)$. Parts (1) and (2) state the precise content of the proof of Drużkowski's "Reduction Theorem" (Theorem 1) in his recent paper [10], although he does not explicitly use the equivalence relation $A \tilde{\cong} D$, and the statement of his Theorem 1 is worded differently. \square

Corollary. Every admissible matrix A is $P3$ -Similar to a *nilpotent* matrix D .

Proposition 6. If $P(i, j)$ is the $n \times n$ permutation matrix obtained by interchanging the i^{th} and j^{th} rows of the $n \times n$ identity matrix I , then every 3-admissible matrix A is *power-3-similar*, by means of $P(i, j)$, to another 3-admissible matrix \tilde{A} , whose i^{th} and j^{th} rows and columns are interchanged from A . (Matrices $Q(i, j)$ with nonzero entries of $P(i, j)$ replaced by ± 1 's also transform 3-admissible matrices to 3-admissible matrices.) Indeed, A and \tilde{A} are permutation-similar iff they are permutation-*power-3-similar*.

Proposition 7. Every 3-admissible $n \times n$ matrix A is *power-3-similar* to another 3-admissible matrix \tilde{A} by means of $S_d = diag[d_1^3, \dots, d_n^3]$. Here, of course, $d_1 \cdots d_n \neq 0$. For example, when $n = 3$,

$$\tilde{A} = \begin{bmatrix} a_{11} (d_1)^2 & \frac{a_{12} (d_2)^3}{d_1} & \frac{a_{13} (d_3)^3}{d_1} \\ \frac{a_{21} (d_1)^3}{d_2} & a_{22} (d_2)^2 & \frac{a_{23} (d_3)^3}{d_2} \\ \frac{a_{31} (d_1)^3}{d_3} & \frac{a_{32} (d_2)^3}{d_3} & a_{33} (d_3)^2 \end{bmatrix}.$$

Here's a 4-dimensional EXAMPLE:

$$\text{If } A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ then } \tilde{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$S_d = \begin{bmatrix} a^{9/2} & 0 & 0 & 0 \\ 0 & a^{9/2} & 0 & 0 \\ 0 & 0 & a^{3/2} & 0 \\ 0 & 0 & 0 & a^{1/2} \end{bmatrix}.$$

Proposition 8. Every 3-admissible 2×2 matrix A has the form

$$\begin{bmatrix} -(ab^3) & a^4 \\ -b^4 & a^3b \end{bmatrix}$$

for some complex numbers a and b ; and each such matrix is 3-admissible.

Proposition 9. Every 3-admissible 2×2 matrix is *P3-Similar* (but not *P2-Similar* unless $a = b$ or $ab = 0$) to the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

by means of any of the nonsingular matrices (with parameters $t \neq 0$ and p_{12})

$$P = \begin{bmatrix} a^3 t^3 & p_{12} \\ b^3 t^3 & \frac{t + p_{12} b^3}{a^3} \end{bmatrix}.$$

Proposition 10. If $c \neq 0$, then $\forall A, cA \not\sim A$. Contrast this with ordinary similarity.

Proposition 11. Every 3-admissible 3×3 matrix is *power-3-similar* to one, or the other, of the following two Jordan Normal Forms:

$$\text{Either } J(1.2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ or } J(2.3) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Proposition 12. The following six 4×4 nilpotent matrices are 3-admissible and mutually inequivalent with respect to *power-3-similarity*. (Integers denote rank and nilpotence index.) Furthermore, except for $N(2.3)$, they each have Property \mathcal{D} of Prop. 5. These are representatives of six *P3S*-equivalence classes in dimension 4.

$$\begin{array}{lll} J(1.2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & J(2.3) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & N(2.3) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ J(2.2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & J(3.4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & N(3.4) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

5 Some Questions to Resolve

1. Is every 3-admissible 4×4 matrix *power-3-similar* to one of the above six?
2. If A is both nilpotent and 3-admissible, then must its **admissibility index** equal its **nilpotence index**? That is, must $\alpha_3(A) = \nu(A)$?
3. If $A \stackrel{\sim}{\sim} B$ and both A and B are nilpotent as well as 3-admissible, then do A and B have the same nilpotence index ?
4. Is every 3-admissible matrix *power-3-similar* to a triangular matrix?
5. Is every 3-admissible matrix *power-3-similar* to a (0,1)-matrix?
6. Is every 3-admissible matrix *power-3-similar* to a nilpotent (0,1)-matrix?
7. How many *P3-Similarity* equivalence classes of (nonzero) 3-admissible matrices exist in each dimension $n \geq 2$? Answer to date: **One** in dim 2; **two** in dim 3; and at least **six** (and I conjecture *only six*) in dim 4. Even if, in some higher dimension, there are *infinitely many* equivalence classes, it is conceivable (even likely) that there are only a finite number of *families* of equivalence classes (in that dimension) and the members of each family are expressible in a reasonable manner by means of parameters or functions which may range through infinitely many values.

6 Some Additional Remarks

1. If $A \stackrel{\sim}{\sim} B$ and both A and B are nilpotent as well as 3-admissible, then A and B do not necessarily behave the same with respect to Property \mathcal{D} of Proposition 5. An example is $N(2.3) \stackrel{\sim}{\sim} D = H'_D(c)$ with

$$c = \begin{bmatrix} c_1 \\ c_2 \\ 1/8\sqrt{3} \\ 1/3\sqrt{3} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 & 8 & 0 \\ 0 & 0 & 8/3 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix D fulfils Proposition 5 for the 3-admissible 4×4 matrix $A = N(2.3)$ listed in Proposition 12. There is no (0,1)-matrix D fulfilling Proposition 5 for $N(2.3)$. In particular, $N(2.3) \neq H'_{N(2.3)}(c)$ for any vector c .

2. The Corollary of Prop. 5 says that every matrix is *P3-Similar* to a nilpotent, but not necessarily to a (0,1)-nilpotent, although this is true in all the examples given here.
3. Note that, although every 3-admissible 2×2 matrix (cf. Prop. 8) is *P3-Similar* to a nilpotent (cf. Prop. 9), not all admissible matrices are themselves nilpotent. In particular, eigenvalues are not preserved by the equivalence relation *P3S* even in dimension 2. Note the *trace* of the 2×2 matrix in Prop. 8.
4. What established mathematical theories or techniques can be brought to bear on the problem of finding “simplest” representatives for 3-admissible matrices in their power-3-similarity classes?

7 Summary of Other Results

Several further relations between the two conjectures **MY** and **JC** have been established in [11, 12, 13, 14, 15, 8, 16, 17]. Some of these results are now briefly described.

1987 [12] (Theorem 1) For polynomial maps $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ satisfying $\det f'(x) \equiv \text{constant} \neq 0$,

(a) \exists a polynomial map $g : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $\forall x \in \mathbf{R}^n, g(f(x)) = x = f(g(x))$
if and only if

(b) $\forall a \in \mathbf{R}^n$, the solution $\Phi(t, x_0, a)$ of the Ważewski DE $\dot{x} = [f'(x)]^{-1}a$, with $x(0) = x_0$, is *polynomial* in both x_0 and t .

(Theorem 1') For polynomial maps satisfying $\det f'(x) \neq 0$ on \mathbf{R}^n ,

(a') \exists a real analytic map $g : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $\forall x \in \mathbf{R}^n, g(f(x)) = x = f(g(x))$
if and only if

(b') $\forall a, x_0 \in \mathbf{R}^n$, the solution $\Phi(t, x_0, a)$ of the Ważewski DE $\dot{x} = [f'(x)]^{-1}a$ is *complete* (exists for all t).

1988 [13] The Markus-Yamabe Conjecture for *2-dimensional polynomial* vector fields is proved true. It remains an open question for polynomial vector fields in all other dimensions $n \geq 3$; and for \mathcal{C}^1 vector fields in dimensions 2 and 3. Barabanov [18] has given a \mathcal{C}^1 -counterexample in dimension 4. Consider the DE $\dot{x} = Ax + b\varphi(\sigma)$, $\sigma = c^*x$, where A is an $n \times n$ matrix, b and c are column vectors, and φ is a scalar function. Assume that $\forall \mu \in (\alpha, \beta)$ the DE with $\varphi(\sigma) = \mu\sigma$ is asymptotically stable. By strengthening the assumptions in Aizerman's Problem, R. E. Kalman conjectured that if $\varphi'(\sigma) \in (\alpha, \beta) \forall \sigma$, then the origin is globally asymptotically stable. Barabanov proved Kalman's conjecture if $\dim n = 3$, and gave an example which shows that DEs exist in dimension $n \geq 4$ which satisfy Kalman's condition but still have a nontrivial periodic solution. This is then also a \mathcal{C}^1 -counterexample to the Markus-Yamabe Conjecture in dimensions ≥ 4 .

1990 [14] A new criterion is given for injectivity of class \mathcal{C}^1 maps f of \mathbf{R}^2 into itself: Namely, $\det f'(x) \neq 0$ on \mathbf{R}^2 and \exists two linearly independent vectors v_i ($i = 1, 2$) in \mathbf{R}^2 such that $0 \notin \text{convex hull of } \{f'(x)v_i \mid x \in \mathbf{R}^2\}$, $i = 1, 2$. This generalizes the earlier (1963) results of Olech [2].

1991 [15] Studies maps of the form $F(x) = x - Q(x)$, where $Q(x)$ is quadratic homogeneous and the jacobian $Q'(x)$ is nilpotent. All such $Q(x)$ have yet to be classified. Explicit formulas are given for the inverses of many families of quadratic maps. For example, the inverse of every *3-dimensional* quadratic mapping $F(x) = x - Q(x)$ can be found by first solving explicitly, $\forall v \in \mathbf{R}^3$, the differential equation

$$dx/dt = v + Q'(x)v + Q'(x)^2v, \quad x(0) = x_0 \in \mathbf{R}^3.$$

Its complete solution can be written explicitly as

$$\begin{aligned}\Phi(t, x_0, v) &= x_0 + \{v + Q'(x_0)v + Q'(x_0)^2v\}t \\ &\quad + \{Q(v) + Q'(v)^2x_0 + Q'(Q(v))x_0 + Q(Q'(v)x_0)\}t^2 \\ &\quad + \{Q'(v)Q(v) + Q'(Q(v))Q'(v)x_0\}t^3 \\ &\quad + Q(Q(v))t^4.\end{aligned}$$

And then

$$F^{-1}(v) = \Phi(1, 0, v) = v + Q(v) + Q'(v)Q(v) + Q(Q(v)).$$

(Note that it would suffice to solve the DE in the simplest case when $x_0 = 0$.) The *4-dimensional* case, as well as several special *5-dimensional* cases are also treated. So DEs of the form $\dot{x} = -F(x)$ are globally asymptotically stable at the origin.

1992 [8] This paper is a survey, but also contains some results not published elsewhere such as the *mean-value formula* for the case of cubic maps of the form $F(x) = x - H(x)$, where $H(tx) \equiv t^3H(x)$ and $H'(x)$ is nilpotent. This was motivated by the following simple situation in the quadratic case. If $F(x)$ is a polynomial map of degree 2, then one can easily verify that it satisfies the **mean-value formula**:

$$F(x) - F(y) \equiv F'\left(\frac{x+y}{2}\right)(x-y).$$

Note that one immediate consequence of this identity is that $\det F'(x) \neq 0 \quad \forall x$ implies that the mapping is injective. So **JC** is true in all dimensions *for quadratic maps!* It seems natural to ask ‘‘What is the corresponding situation for cubic maps?’’ Here’s the answer: Each cubic-homogeneous map H of \mathbf{C}^n into itself gives rise to a unique matrix-valued bilinear mapping $B(x, y)$, namely,

$$B(x, y) := H'\left(\frac{x+y}{2}\right) - H'\left(\frac{x-y}{2}\right),$$

such that

1. $H'(x) \equiv B(x, x)$,
2. $B(x, y) \equiv B(y, x)$, and
3. $B(x, y)z \equiv B(x, z)y$.
4. Also $H(x) \equiv \frac{1}{3}H'(x)x \equiv \frac{1}{3}B(x, x)x$.

And then, if $N(x, y) := \frac{1}{3}\{B(x, x) + B(x, y) + B(y, y)\}$, one obtains

$$F(x) - F(y) = \{I - N(x, y)\}(x - y).$$

Corollary: The mapping $F(x) = x - H(x)$ is injective if and only if, $\forall a, b \in \mathbf{C}^n$, $(a - b)$ is not an eigenvector of $B(a, b)$ corresponding to the eigenvalue 1.

Notice that if $B(x, y)$ is nilpotent for all x and y , then this condition is certainly satisfied because all of its eigenvalues are zero. Most examples seem to be of this type, but a 5-dimensional example is given in [8] where $B(a, b)$ has eigenvalue 1 for some $a \neq b$, even though $B(x, x) \equiv H'(x)$ is nilpotent $\forall x \in \mathbf{C}^n$.

1992 [16] Generalizes the above bilinear B -matrix to the multilinear case and gives new proofs of the Connell-Zweibel results in [19, 20], and relates these results to differential equations. This yields a large class of nonlinear (polynomial) vector fields which have the origin as a globally asymptotically stable rest point. Given a homogeneous polynomial matrix $M(x)$, which is both nilpotent and power-exact (i.e., each of its powers is the jacobian of some vector map), then the map

$$F(x) := x + \frac{M(x)x}{(m+1)} + \frac{M(x)^2x}{(2m+1)} + \cdots + \frac{M(x)^r x}{(rm+1)}$$

is bijective with a polynomial inverse. Furthermore, a new short proof is given that if M and M^2 are both exact, then M^k is also exact for each $k > 2$. Once again the Ważewski differential equation for these maps is solved in order to compute explicit formulas for their inverses. Adjmagbo and van den Essen [21] obtain invertibility of these Connell-Zweibel maps $F(x)$ by means of their “Eulerian Operators”. One further result in this paper (**Prop 11**):

If $H(x)$ is a homogeneous of degree $(m+1)$ polynomial mapping of \mathbf{R}^n (or \mathbf{C}^n) into itself whose jacobian $H'(x)$ is power-exact, then each iterate of H can be expressed as a power of $H'(x)$ by the formula

$$H^{[k]}(x) = C(m, k)H'(x)^{p(m, k)}x,$$

where $C(m, k)$ is a scalar, $p(m, k)$ is an integer $\geq k$, and $H^{[k]}(x)$ denotes the k^{th} iterate of H : Namely, $H^{[1]}(x) \equiv H(x)$ and $H^{[k]}(x) \equiv H^{[k-1]}(H(x))$. If $H'(x)$ is also *nilpotent*, then there is an integer $k \leq n$ such that $H^{[k]}(x) \equiv 0$. Furthermore, whether $H'(x)$ is nilpotent or not, one has the recursion relations

$$C(m, k) = \frac{C(m, k-1)}{(m+1)^{1+mp(m, k-1)}}, \quad \text{where } p(m, k) = (m+1)p(m, k-1) + 1.$$

1993 [17] This is a survey of the relationships between **JC** and the Markus-Yamabe Conjecture concerning global asymptotic stability.

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