Recent Progress on

The Markus-Yamabe Global Asymptotic Stability Conjecture

&

The Strong Jacobian Conjecture

by

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The Markus-Yamabe global asymptotic stability conjecture on $\mathbb{R}^2$ has been pretty much completely solved (by the end of 1993); and counterexamples in higher dimensions have now appeared (November-December 1995). The main relevant papers are as follows:

1. For polynomial vector fields on $\mathbb{R}^2$ by Meisters & Olech in **Analyse mathématique et applications, Contributions en l'honneur de Jacques-Louis Lions**, Gauthier-Villars, Paris 1988, pages 373-381; **MR 90b**:58135. ⟨`gmeister@math.unl.edu`⟩ & ⟨`olech@impan.gov.pl`⟩.

2. For $\mathbf{C}^1$ vector fields on $\mathbb{R}^2$ by R. Feßler in **Annales Polonici Mathematici 62** (1995) 45–75. And also by C. Gutierrez in **Ann. Inst. H. Poincaré Anal. Non Linéaire** (no. 6) **12** (1995). ⟨`fessler@math.unibas.ch`⟩ & ⟨`gutp@impa.br`⟩. Both were presented at Marco Sabatini's Conference held at Trento, Italy, in September 1993. ⟨`sabatini@itnvax.science.unitn.it`⟩.

3. Counterexample for analytic vector fields in $\mathbb{R}^4$ by Josep Bernat and Jaume Llibre to appear in **Dynamics of Continuous, Discrete and Impulsive Systems**. ⟨`IMATO@cc.uab.es`⟩. They present in clear detail a 4-dimensional analytic system $\dot{x} = F(x)$ with a non-trivial *periodic* orbit (hence bounded away from zero and infinity), whose Jacobian matrix $F'(x)$ is **Hurwitz**: The real parts of all its eigenvalues are negative.

4. Characterization of linearizable dilations of analytic maps by Bo Deng in, *Analytic Conjugation, Global Attractor, and the Jacobian Conjecture*, 1995 Preprint, 4 pages, University of Nebraska-Lincoln, U.S.A. ⟨`bdeng@ode.unl.edu`⟩. Bo proved that the dilation λf of an analytic map f of $\mathbb{C}^n$ into itself with $f(0) = 0$, $f'(0) = I$, and $|\lambda| > 1$, has an *analytic* conjugation to its linear part λx if and only if f is a *holomorphism* (i.e., a holomorphic automorphism) of $\mathbb{C}^n$ and $x = 0$ is a global attractor for the inverse of λf. This is a Corollary of the more general **Theorem:** Suppose $F : \mathbb{C}^n \to \mathbb{C}^n$ is an analytic map, $F(0) = 0$, $\det F'(0) \neq 0$, and all eigenvalues of $F'(0)$ are in the open unit disk and not resonant; then F has an *analytic* conjugation to its linear part $F'(0)$ iff F is an analytic automorphism of $\mathbb{C}^n$ and the fixed point $x = 0$ is a global attractor, i.e., $\forall x \in \mathbb{C}^n$, $F^k(x) := F \circ F^{k-1}(x) \to 0$ as $k \to \infty$. Compare with the theorem in the appendix of the 1988 paper (listed in Meisters' PolyMapBib) by Rosay & Rudin on holomorphic maps. (The latter was kindly brought to our attention by Franc Forstneric, January 10, 1996.)

5. Counterexample for polynomial vector fields in $\mathbb{R}^3$ jointly by Anna Cima, Arno van den Essen, Armengol Gasull, Engelbert Hubbers, and Francesc Mañosas in *A polynomial counterexample to the Markus-Yamabe Conjecture* preprint November 1995. ⟨`hubbers@sci.kun.nl`⟩. This time there are no periodic orbits, but rather orbits which escape to infinity. Is this the general situation for polynomial vector fields satisfying the Markus-Yamabe Hypothesis? That is, do all orbits of such polynomial systems either tend to zero or infinity? I think the authors of this paper agree that the beautiful and elegant paper by Bo Deng ([4] above) played a crucial role in their paper.

6. Counterexample in $\mathbb{R}^2$ to **The Strong Jacobian Conjecture** by Serguey Pinchuk (Pinčuk) in May of 1994 [**Math. Z. 217** (1994) 1–4]. He gave a beautiful example of a non-injective polynomial mapping $F = (p, q)$ of $\mathbb{R}^2$ into itself, of degree$(p, q) = (10, 25)$, but whose Jacobian determinant is everywhere positive on $\mathbb{R}^2$. The more famous **Jacobian Conjecture of O.-H. Keller** remains open: A Polynomial map of $\mathbb{C}^n$ [or of $\mathbb{R}^n$] into itself, for $n > 1$, whose Jacobian determinant is *constant* on $\mathbb{C}^n$ [or on $\mathbb{R}^n$, respectively], is necessarily **injective** (and also surjective with a polynomial inverse). Another name for **The Strong Jacobian Conjecture** is *The Real Jacobian Conjecture*.

An extensive bibliography including all of the above, and much more, can be found on the World-Wide-Web Page of G. H. Meisters at the URL `http://www.math.unl.edu/~gmeister/`; or simply e-mail one of the people mentioned above, or Arno van den Essen at ⟨`essen@sci.kun.nl`⟩.