A New Attempt to Prove The Jacobian Conjecture of Ott-Heinrich Keller

Ends The Markus-Yamabe Conjecture Instead

MYC 1960–1995

Killed by Serendipity!

R. I. P.

Gary H. Meisters
http://www.math.unl.edu/∼gmeister/

October 1996

NOTATION

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ denotes a column vector in } \mathbb{C}^n$$

$$\operatorname{diag}(x) := \begin{pmatrix} x_1 & & & & \\ & x_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot & \\ & & & & & x_n \end{pmatrix}$$

$$\text{Define } \mathbf{1} := \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \text{ so that } \operatorname{diag}(x)\mathbf{1} = x.$$

Keller's Jacobian Conjecture (1939)

Does $F : \mathbb{C}^n \to \mathbb{C}^n$ polynomial & $\det F'(x) \equiv 1$
$\Rightarrow F$ is bijective with polynomial inverse ?

Ott-Heinrich Keller [1906–1990]

REDUCTIONS: It suffices to prove

1. **injectivity** of F
   [Białynicki-Birula & Rosenlicht (1962); and Rudin (1995)]
2. for **cubic-homogeneous** maps
   $F(x) := x - H(x)$ with $H(tx) = t^3 H(x)$
   [Yagzhev; Bass, Connell & Wright (c. 1980)]
3. or merely for **cubic-linear** maps
   $F_A(x) := x - H_A(x)$ where
   $H_A(x) := [\mathrm{diag}(Ax)]^3 \mathbf{1} = [\mathrm{diag}(Ax)]^2 Ax,$
   for some $n \times n$ matrix A. [Drużkowski (1983)]

Dfn: Call matrix A **admissible** if $\det F'_A(x) \equiv 1$.

Dfn: Call matrix A **good** if F_A is injective.

Theorem: Injectivity of $F \Rightarrow \det F'(x) \equiv 1$.
In particular, *every good matrix is admissible.*

Proof: $I = x' = [G(F(x))]' = G'(F(x))F'(x)$.
So $\det I = 1 = \det G'(F(x)) \det F'(x)$. □

JC is converse: *Is every admissible matrix good?*

The Markus-Yamabe Conjecture on Global Asymptotic Stability

If $\mathcal{C}^1$ map $f : \mathbb{R}^n \to \mathbb{R}^n$ has a fixed point at the origin, $f(0) = 0$, and if its Jacobian matrix $f'(x)$ is **stable** $\forall x \in \mathbb{R}^n$, then 0 is a **global attractor** of the system

$$\frac{dx}{dt} = f(x). \qquad (1960)$$

1986: MYC proved for *polynomial* vector fields on $\mathbb{R}^2$.
[Meisters & Olech]

1993: MYC proved for class $\mathcal{C}^1$ vector fields on $\mathbb{R}^2$.
[Independently by Robert Feßler and Carlos Gutierrez]

1994: $\exists$ a $\mathcal{C}^\infty$ MY-system on $\mathbb{R}^4$ with a *periodic orbit*.
[Bernat & Llibre inspired by a 1988 paper of Barabanov.]

Mar 1994: *Polynomial examples* of $h(s, sF(x)) = sh(s,x)$.
[Found by Meisters at the request of Deng & Zampieri.]

Sep 1994: $\exists$ dim 4 deg 3 $F = X - H$ with h_s *only analytic*.
[van den Essen won $100 from Meisters for this example.]

May 1995: *Characterization of linearizable analytic maps*.
[Bo Deng re-discovered this 1988 result of Rosay & Rudin.]

Nov 1995: $\exists$ dim 4 deg 5 $F = X - H$ whose h_s is *not analytic*.
[van den Essen & Hubbers apply Bo Deng's criterion.] DMZ False!

Dec 1995: $\exists$ *polynomial* MY-system on $\mathbb{R}^3$ with $x(t) \to \infty$.
[Cima, van den Essen, Gasull, Hubbers & Mañosas] MYC FALSE!

Aug 1996: A dim 15 cubic-linear F_A has *polynomial* h_s.
But *non-poly* (analytic) h_s occur. [Gorni & Zampieri.]

The Poincaré-Siegel Theory of Non-Resonant Eigenvalues

The idea of conjugating an analytic map *locally* to its linear part, in a neighborhood of a fixed point, was introduced by Poincaré in his doctoral thesis; later (c. 1942) C. L. Siegel studied local convergence of the formal power series for the conjugation function $h(x)$.

This theory of **non-resonant linearization** shows that dilated polyomorphisms $sf(x) = sx - sg(x)$ are *analytically* conjugate to sx, at least *locally*, if $|s| \neq 1$. This conjugation is *global* and *polynomial*, for *all* cubic-homogeneous examples I worked out in March 1994 and presented at Curaçao on July 4, 1994.

Resonance of order $|m|$. The n-tuple $\sigma = (s_1, \ldots, s_n)$ of eigenvalues of a linear map L, such as the derivative map $x \mapsto f'(0)x$ at the fixed-point $x = 0$ of a mapping $f(x)$, is said to be **resonant** if one of these eigenvalues (say s_k) satisfies: $s_k = \sigma^m = s_1^{m_1} \cdots s_n^{m_n}$, for integers $m_j \geq 0$ with $|m| = m_1 + \cdots + m_n \geq 2$.

Poincaré. For $f : \mathbb{C}^n \to \mathbb{C}^n$ analytic at $x = 0$, $f(0) = 0$, and eigenvalues $(s_1, \ldots, s_n)$ of $f'(0)$ **not resonant**, there is a formal power series $h(x)$ so that *locally* at 0

$$h \circ f \circ h^{-1}(x) = f'(0)x.$$

Siegel's Theorem and Dilated Polyomorphisms

Siegel's Thm. Let f be analytic at $x = 0$ with $f(0) = 0$. If the eigenvalues $\sigma = (s_1, \ldots, s_n)$ of $f'(0)$ are of **multiplicity-type** (C, ν), for constants $C > 0$ and $\nu > 0$, in the sense that

$$|s_j - \sigma^m| \geq C/|m|^\nu \tag{1}$$

$\forall m = (m_1, \ldots, m_n)$ with $|m| \geq 2$ and $1 \leq j \leq n$, then the formal power series for h and h^{-1} which occur in Poincaré's Theorem *converge* near $x = 0$.

The Case of Dilated Polyomorphisms. For each complex s, the eigenvalues of the linear map $x \mapsto sf'(0)$ are all equal to s. They are resonant of order $|m| \geq 2$ iff

$$s = s^{m_1} \cdots s^{m_n} = s^{|m|},$$

or $s^{|m|-1} = 1$, or s is a root of unity.

If $0 \neq |s| \neq 1$, then Siegel's inequality (1) holds for

$$C = \left||s| - |s|^2\right|$$

and any $\nu \geq 1$, because $\sigma^m = s^{|m|}$ and

$$\left|s - s^{|m|}\right| \geq \left||s| - |s|^{|m|}\right| \geq \left||s| - |s|^2\right| > C/2 \geq C/|m|^\nu.$$

The Schröder Map $h_s(x)$
& The DMZ Conjecture

Deng & Meisters & Zampieri, March 1994:

When does the Poincaré-Siegel Method
of Local Linearization at a Fixed-Point
extend to GLOBAL LINEARIZATION ?

This leads to the end of MYC & new questions!

To each admissible A there is a power series in x

$$h_s(x) := h(s,x) = x + \sum_{k=1} c_k(s) g_k(x) \qquad (2)$$

with coefficients in $\mathbb{Z}(s)$ such that, for $s^k \neq 1$,

$$h(s, sF_A(x)) = sh(s,x) \quad \forall\, x \in \mathbb{C}^n. \qquad (3)$$

Thus, if h_s^{-1} exists, h_s *conjugates* sF to sI:

$$\boxed{h_s \circ sF_A \circ {h_s}^{-1} = sI.}$$

The first few terms of the power series for $h_s(x)$ are:

$$h_s(x) = x + \frac{H_A(x)}{(s^2-1)} + \frac{s^2 {H_A}'(x) H_A(x)}{(s^2-1)(s^4-1)}$$

$$+ \frac{s^2(1+s^4){H_A}'(H_A(x))x}{(s^2-1)(s^4-1)(s^6-1)}$$

$$+ \frac{s^2(1+s^2)(1-s^2+3s^4-s^6+s^8)H_A(H_A(x))}{(s^2-1)(s^4-1)(s^6-1)(s^8-1)} + \cdots$$

If F is injective, $F_A^{-1}(x) = \lim_{t\to\infty} \frac{1}{t} h(t,tx)$.

For which admissible A is h_s a polyomorphism?

For which admissible A is h_s a holomorphism?

An F_A whose h_s is polynomial

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad F_A = \begin{pmatrix} x - y^3 \\ y - z^3 \\ z \end{pmatrix}$$

$$F_A^{-1} = \begin{pmatrix} x + y^3 + 3\,y^2\,z^3 + 3\,y\,z^6 + z^9 \\ y + z^3 \\ z \end{pmatrix}$$

$$h_s = \begin{pmatrix} \left\{ \begin{array}{c} x + \frac{y^3}{s^2-1} \;+\; \frac{3\,s^2\,y^2\,z^3}{(s^2-1)(s^4-1)} \\ + \frac{3\,s^2\left(s^4+1\right)y\,z^6}{(s^2-1)(s^4-1)(s^6-1)} \\ + \frac{s^2\left(1-s^2+3s^4-s^6+s^8\right)z^9}{(s^2-1)^2(s^6-1)(s^8-1)} \end{array} \right\} \\ y + \frac{z^3}{s^2-1} \\ z \end{pmatrix}$$

$$h_s^{-1} = \begin{pmatrix} \left\{ \begin{array}{c} x + \frac{y^3}{s^2-1} + \frac{3y^2z^3}{(s^2-1)(s^4-1)} - \\ \frac{3\,(s^2+1)\,yz^6}{(s^2-1)(s^4-1)(s^6-1)} + \frac{z^9}{(s^2-1)^3(s^8-1)} \end{array} \right\} \\ y - \frac{z^3}{s^2-1} \\ z \end{pmatrix}$$

This and 23 other examples, *all polynomial*, were presented at the Curaçao Conference, July 4, 1994.

The \$100 Counterexample: A 4D cubic-homogeneous polyomorphism F whose complex s-dilations sF are not globally polyomorphic to sx.

<u>Arno van den Essen, September, 1994</u>:

Let $p(x) = x_1x_3 + x_2x_4$.

Then the map $F = I - H$ is:

$$\begin{aligned} y_1 &= x_1 + p(x)x_4 \\ y_2 &= x_2 - p(x)x_3 \\ y_3 &= x_3 + x_4{}^3 \\ y_4 &= x_4 \end{aligned}$$

But sF *is globally analytically conjugate* to its linear part sx. So at this point the DMZ-CONJECTURE was still open. What was needed was a *criterion* to help find maps F whose dilations are *not even analytically* conjugate to their linear part.

Bo Deng found such a *criterion* in the Spring of 1995.

Bo Deng's Characterization of Globally Linearizable Analytic Maps

Bo Deng's 4-page preprint, May, 1995:

Analytic Conjugation, Global Attractor, and the Jacobian Conjecture.

Bo's Lemma: A dilation λf of an analytic map f of $\mathbb{C}^n$ into itself with $f(0) = 0$, $f'(0) = I$, and $|\lambda| > 1$, has a *global analytic conjugation* to its linear part λx if and only if f is a *holomorphism* of $\mathbb{C}^n$ and $x = 0$ is a global attractor for the inverse of λf.

More Generally:

If F is an analytic map of $\mathbb{C}^n$ into itself, $F(0) = 0$, $\det F'(0) \neq 0$, and all eigenvalues λ of $F'(0)$ are **non-resonant** and satisfy $|\lambda| < 1$; then F is *analytically* conjugate to $F'(0)$ *iff* F is a *holomorphism* of $\mathbb{C}^n$ and 0 is a **global attractor**: i.e., $\forall x \in \mathbb{C}^n$, $F^k(x) := F \circ F^{k-1}(x) \to 0$, as $k \to \infty$.

Not New:

Franc Forstneric brought to our attention in Jan 96, that this latter result is in a 1988 paper by Rosay & Rudin: *Holomorphic Maps from* $\mathbb{C}^n$ *to* $\mathbb{C}^n$, TAMS.

But it was Bo's Lemma that led van den Essen and his colleagues to find counterexamples to the DMZ-C & the MY-C.

Counterexample to the DMZ Conjecture: A 4D, degree 5, polyomorphism F whose complex s-dilations sF are not globally holomorphic to sx.

This counterexample was found, with the help of Bo's Lemma, by van den Essen & Hubbers, November 1995:

Define $p(x) = x_1x_3 + x_2x_4$
and let m be an odd integer ≥ 3.

Then the map $F = I - H$ defined by

$$\begin{aligned} y_1 &= x_1 + p(x)^2 x_4 \\ y_2 &= x_2 - p(x)^2 x_3 \\ y_3 &= x_3 + {x_4}^m \\ y_4 &= x_4 \end{aligned}$$

is a counterexample to the **DMZ** Conjecture.

But no counterexamples of the **cubic-linear** form

$$F_A(x) = x - [\mathrm{diag}(Ax)]^2 Ax$$

with *admissible* A, $\det F'_A(x) \equiv 1$, have been found!

A 3D Polynomial Counterexample to the Markus-Yamabe Conjecture

Cima, van den Essen, Gasull, Hubbers & Mañosas · Dec 1995

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_3(x_1 + x_2x_3)^2 \\ \dot{x}_2 &= -x_2 - (x_1 + x_2x_3)^2 \\ \dot{x}_3 &= -x_3 \end{aligned}$$

Zero is a stationary point & the Jacobian is:

$$f\,'(x) = \begin{pmatrix} -1 + 2dx_3 & 2dx_3{}^2 & d^2 + 2dx_2x_3 \\ -2d & -1 - 2dx_3 & -2dx_2 \\ 0 & 0 & -1 \end{pmatrix}$$

where $d = (x_1 + x_2x_3)$.

The characteristic equation of $f\,'(x)$ is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0,$$

so $\forall x \in \mathbb{R}^3$ all eigenvalues of $f\,'(x)$ are -1 .

But the system has the unbounded solution:

$$\begin{aligned} x_1 &= 18\,e^t \\ x_2 &= -12e^{2t} \\ x_3 &= e^{-t} \end{aligned}$$

2D Solutions of the 3D Polynomial Counterexample to the Markus-Yamabe Conjecture

$$\begin{aligned}
\dot{x}_1 &= -x_1 + x_3(x_1 + x_2 x_3)^2 \\
\dot{x}_2 &= -x_2 - (x_1 + x_2 x_3)^2 \\
\dot{x}_3 &= -x_3
\end{aligned}$$

When $c_3 = 0$ the general solution is:

$$\begin{aligned}
x_1(t, c_1, c_2, 0) &= c_1 e^{-t} \\
x_2(t, c_1, c_2, 0) &= \left(c_2 - {c_1}^2\right) e^{-t} + {c_1}^2 e^{-2t} \\
x_3(t, c_1, c_2, 0) &= 0
\end{aligned}$$

A 4D analytic MY-system with a periodic orbit

Bernat & Llibre 1994 via Barabanov's 1988-paper

$$(\text{CS}\star)\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_4 \\ \dot{x}_3 = x_1 - 2x_4 - \frac{9131}{900}\psi(x_4) \\ \dot{x}_4 = x_1 + x_3 - x_4 - \frac{1837}{180}\psi(x_4) \end{cases}$$

Bernat & Llibre prove $\exists$ a $\mathcal{C}^1$ function ψ such that this system (CS$\star$) is a MY-system with a (nonzero) *periodic* orbit. They remark that ψ can be chosen $\mathcal{C}^r$, for any $r \geq 1$, or $\mathcal{C}^\infty$, or even *analytic*.

Until a *polynomial* MY-system is found with a *periodic* orbit, this example remains interesting.

Could it be true that

Every <u>*forward-bounded*</u> *solution of a polynomial* MY-*system tends to the equilibrium-point 0*?

A Restricted Form of the Markus-Yamabe Conjecture $\forall n$ is Equivalent to Keller's Jacobian Conjecture $\forall n$

(RMYC): *Every system of the special form*

$$\frac{dx}{dt} = G(x) = c - x + H(x),$$

with $H(tx) = t^3 H(x)$ and $H^{'}(x)$ nilpotent $\forall x \in \mathbb{C}^n$, *has at most one rest point* x_0.

Note that all eigenvalues of $G^{'}(x)$ are -1.

By reduction-of-degree we see that JC is false iff there is a polynomial map $F(x) = x - H(x)$, with $H(tx) = t^3 H(x)$ and $H^{'}(x)$ nilpotent $\forall x \in \mathbb{C}^n$, and yet the system

$$\frac{dx}{dt} = G(x) = F(x_1) - F(x) = c - x + H(x)$$

has *two distinct rest points* x_1, x_2.
(That is, $F(x)$ is not injective.)

The 3D polynomial counterexample to MYC, found by Anna Cima et al, *is not of this form*!

A Question Equivalent to JC

The Phantom Eigenvector Problem:

Do 'Bad' Matrices Exist?

Each $n \times n$ complex matrix A defines a bilinear map

$$\mathcal{B}_A(x, y) := 3\,\mathrm{diag}(Ax)\,\mathrm{diag}(Ay)A.$$

$$\mathcal{B}_A : \mathbb{C}^n \times \mathbb{C}^n \to \mathcal{M}_n(\mathbb{C}).$$

Definition: A matrix A is called **bad** if it is both:

1. **admissible:** $\forall x \in \mathbb{C}^n\,,\ \ \mathcal{B}_A(x, x)^n = 0$, and also
2. **odd:** $\exists\, x \neq y\,,\ \ \mathcal{B}_A(x, y)(x - y) = (x - y)$.

WANTED: A BAD MATRIX

\$\$\$\$\$\$\$ REWARD \$\$\$\$\$\$\$

(NO PHOTO AVAILABLE)

The Bilinear $\mathcal{B}$-matrix $\mathcal{B}_A(x, y)$

Each cubic-homogeneous map $H : \mathbb{C}^n \to \mathbb{C}^n$ defines a unique matrix-valued bilinear map $(x, y) \mapsto \mathcal{B}(x, y)$ such that $\forall\ x,\ y,\ z \in \mathbb{C}^n$

1. $\mathcal{B}(x, x) = H'(x)$,
2. $\mathcal{B}(x, y) = \mathcal{B}(y, x)$,
3. $\mathcal{B}(x, y)z = \mathcal{B}(x, z)y$, and
4. $H(x) = (1/3)H'(x)x = (1/3)\mathcal{B}(x, x)x$.

Namely,

$$\mathcal{B}(x, y) := \frac{1}{4}\{H'(x + y) - H'(x - y)\}.$$

If we define the mapping $F(x) := x - H(x)$, then $F'(x) = I - H'(x)$, and

$$\boxed{\det F'(x) \equiv 1 \text{ iff } H'(x)^n \equiv 0.}$$

If $H_A(x) := [\text{diag}(Ax)]^3\mathbf{1}$ is defined by a matrix A, then $F_A(x) = x - H_A(x)$ is **cubic-linear** and

$$\mathcal{B}_A(x, y) := 3[\text{diag}(Ax)][\text{diag}Ay)]A.$$

Call A **beautiful** if $[\mathcal{B}_A(x, y)]^n = 0 \quad \forall x,\ y \in \mathbb{C}^n$. *Beautiful* matrices A are obviously *admissible*. But, they cannot be odd! (All eigenvalues are zero.) In our *Ames paper* (Oct 1991), we proved that A is *good* iff it is *not odd*: So

Beautiful matrices A are *good*!

A 15×15 **admissible** matrix A that is **not bad**

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 2 & -2 & -2 & -2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\
0 & 0 & 2 & 0 & 2 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & -2 & 4 & 0 & 0 & 0 & -2 & 2 & 2 & 2 & 0 & 0 & 0 & -2 \\
-2 & 0 & -2 & 4 & 0 & 0 & 0 & -2 & 2 & 2 & 2 & 0 & 0 & 0 & -2 \\
0 & -2 & -2 & 4 & 0 & 0 & 0 & -2 & 2 & 2 & 2 & 0 & 0 & 0 & -2 \\
-2 & 0 & 2 & 0 & 2 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\
-2 & 0 & 0 & 4 & 2 & -2 & -2 & -2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\
0 & -2 & 0 & 4 & 2 & -2 & -2 & -2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\
-2 & 0 & -2 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\
0 & -2 & 2 & -4 & 0 & 0 & 0 & 2 & -2 & -2 & -2 & 0 & 0 & 0 & 2 \\
0 & -2 & 0 & -4 & -2 & 2 & 2 & 2 & 0 & 0 & -2 & 0 & 0 & -2 & 0 \\
-2 & -2 & -2 & 4 & 0 & 0 & 0 & -2 & 2 & 2 & 2 & 0 & 0 & 0 & -2 \\
-2 & -2 & 0 & 4 & 2 & -2 & -2 & -2 & 0 & 0 & 2 & 0 & 0 & 2 & 0
\end{pmatrix}$$

In fact, it is **good** but **ugly** (not beautiful):

A has rank 5, nilpotent-index 2, and $[\mathcal{B}_A(x,x)]^5 = 0$.

$$\boxed{F_A \text{ is injective } (A \text{ is good}); \text{ but } [\mathcal{B}_A(e_1,e_2)]^{15} \neq 0 \ (A \text{ is ugly}).}$$

That $[\mathcal{B}_A(x,x)]^5 = 0 \ \forall x \in \mathbb{C}^{15}$ is harder to show:
It suffices to show the 13×13 lower-right block of $\mathcal{B}_A(x,x)$ has nilpotence-index 4. (A computer check).

$$\det [\, \lambda I - \mathcal{B}_A(x,y)\,] = \lambda^{15} + 576(x_1y_2 - x_2y_1)^2 \,\lambda^{13}.$$

In August 1996 Gorni & Zampieri proved that:
The Schröder map $h_s(x)$ for the associated mapping $F_A(x) = x - H_A(x)$ is *polynomial*.

$$\boxed{h_s \circ sF_A \circ {h_s}^{-1} = sI.}$$

All 2×2 Admissible Matrices

have the dyadic form

$$A = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} -b^3 & a^3 \end{pmatrix} = \begin{pmatrix} -ab^3 & a^4 \\ -b^4 & a^3 b \end{pmatrix},$$

for some complex numbers a and b.

Every such A is *beautiful*; i.e.,

$$[\mathcal{B}_A(x, y)]^2 = 0 \ \ \forall x,\ y \in \mathbb{C}^2,$$

because $\mathcal{B}_A(x, y) =$

$$3(a^3 x_2 - b^3 x_1)(a^3 y_2 - b^3 y_1) \begin{pmatrix} a^3 \\ b^3 \end{pmatrix} \begin{pmatrix} -b^3 & a^3 \end{pmatrix}.$$

Note also that

1. $\text{trace}(A) = ab(a^2 - b^2)$.
2. $\det(A) = 0$.
3. $\text{CharPoly}(A) = \lambda^2 - ab(a^2 - b^2)\lambda$.
4. And the matrix A is *good*: That is, the mapping

$$\begin{aligned} F_A(x) &= x - (1/3)\mathcal{B}_A(x, x)x \\ &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (a^3 x_2 - b^3 x_1)^3 \begin{pmatrix} a^3 \\ b^3 \end{pmatrix} \end{aligned}$$

is injective.

Another Little Question

Each $n \times n$ complex matrix A determines a **binary relation** $x \boxed{A} y$ on $\mathbb{C}^n$ by the eq.

$$\mathcal{B}_A(x, y)(x - y) = (x - y),$$

which uses the bilinear map

$$\mathcal{B}_A(x, y) = 3\,\mathrm{diag}(Ax)\,\mathrm{diag}(Ay)\,A.$$

Properties of $x \boxed{A} y$:

r : $x \boxed{A} x$.

s : $x \boxed{A} y \;\Rightarrow y \boxed{A} x$.

t : Does $x \boxed{A} y \;\&\; y \boxed{A} z \;\overset{?}{\Rightarrow}\; x \boxed{A} z$.

$\boxed{\text{Is } x \boxed{A} y \text{ an } \textbf{equivalence relation}?}$

If A is admissible & JC is true, then "YES"!
(It's *equality*! $x \boxed{A} y$ *if and only if* $x = y$.)

Summary

Beautiful A $\mathcal{B}_A(x,y)$ nilpotent	$\overset{\overset{\text{not}}{\longleftarrow}}{\longrightarrow}$	**Admissible** A $\mathcal{B}_A(x,x)$ nilpotent
$\Downarrow \quad \not\Uparrow$		$\Downarrow ? \quad \Uparrow$
A **is Not Odd** $\mathcal{B}_A(x,y)(x-y) = (x-y)$ $\Rightarrow \quad x = y$	$\Longleftrightarrow$	A **is Good** $F_A(x)$ is injective

$$\mathcal{B}_A(x,y) := 3\,\mathrm{diag}(Ax)\,\mathrm{diag}(Ay)\,A.$$

$$F_A(x) := x - [\mathrm{diag}(Ax)]^3\mathbf{1} = x - \tfrac{1}{3}\mathcal{B}_A(x,x)x.$$

$$x\,\boxed{A}\,y :\equiv: \mathcal{B}_A(x,y)(x-y) = (x-y)$$

Is this last an equivalence relation on $\mathbb{C}^n$?

For each A? Or at least for each admissible A?

The Cubic-Similarity Equivalence Relation $A \overset{cs}{\approx} D$

Call matrices A and D **cubic-similar** if there is an invertible matrix P such that

$$[\mathrm{diag}(APx)]^3\mathbf{1} = P[\mathrm{diag}(Dx)]^3\mathbf{1} \ \ \forall x \in \mathbb{C}^n,$$

where

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Representative Matrices of the Cubic-Similarity Equivalence Classes

The Real Jacobian Conjecture

$F : \mathbb{R}^n \to \mathbb{R}^n$ poly & $\det[F'(x)] \neq 0$ on $\mathbb{R}^n$

$\Rightarrow F$ is bijective with analytic inverse.

Example: The map $x \mapsto x + x^3 \in \mathbb{R}^1$ has Jacobian $1 + 3x^2 \geq 1$ and is bijective with analytic inverse.

Counterexample (Serguey Pinchuk May 1994):

Define the polynomial map $\langle x, y\rangle \mapsto \langle p, q\rangle$ as follows:

$$\begin{aligned}
t &= xy - 1, \\
h &= t(xt+1), \\
f &= ((h+1)/x)(xt+1)^2, \\
p(x,y) &= f + h, \\
Q &= -t^2 - 6th(h+1), \\
u &= 170fh + 91h^2 + 195fh^2 + 69h^3 \\
&\quad +75h^3 f + (75/4)h^4, \\
q(x,y) &= Q - u.
\end{aligned}$$

Then the Jacobian J is a sum of squares and $\neq 0$:

$$J \equiv \partial(p,q)/\partial(x,y) = t^2 + (t + (13+15h)f)^2 + f^2.$$

$J = 0 \Rightarrow t = 0$ & $f = 0$; then $h = 0$ & $f = 1/x \neq 0$.

Yet, the map is not injective because

$$\langle p, q\rangle(1,0) = \langle p, q\rangle(-1,-2) = \langle 0, -1\rangle$$

Q.E.D.

Pinchuk's Counterexample to RJC Written Explicitly

Degree$(p, q) = (10, 25)$

$$
\begin{aligned}
p(x,y) \;=\;& (1 - x + x^2y) \\
& (-x + y - 2xy + 3x^2y + 2x^2y^2 \\
& -3x^3y^2 + x^4y^3)
\end{aligned}
$$

$$
\begin{aligned}
q(x,y) \;=\;& ((-1 + xy)(-33 + 244x - 398x^2 \\
& +536x^3 - 645x^4 + 300x^5 \\
& -200y + 623xy - 1424x^2y \\
& +3386x^3y - 5980x^4y + 6315x^5y \\
& -2700x^6y - 180xy^2 - 515x^2y^2 \\
& +4516x^3y^2 - 14520x^4y^2 + 26360x^5y^2 \\
& -26145x^6y^2 + 10800x^7y^2 - 300x^2y^3 \\
& +1725x^3y^3 - 10236x^4y^3 + 33224x^5y^3 \\
& -60860x^6y^3 + 60375x^7y^3 - 25200x^8y^3 \\
& -1800x^4y^4 + 11400x^5y^4 - 40442x^6y^4 \\
& +80680x^7y^4 - 85575x^8y^4 + 37800x^9y^4 \\
& -4500x^6y^5 + 24750x^7y^5 - 62036x^8y^5 \\
& +76545x^9y^5 - 37800x^{10}y^5 - 6000x^8y^6 \\
& +25800x^9y^6 - 42315x^{10}y^6 + 25200x^{11}y^6 \\
& -4500x^{10}y^7 + 13245x^{11}y^7 - 10800x^{12}y^7 \\
& -1800x^{12}y^8 + 2700x^{13}y^8 - 300x^{14}y^9))/4.
\end{aligned}
$$

An ANALYTIC Example:

Admissible but not Good

Define the mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$u(x, y) = \sqrt{2}\exp(x/2)\cos(y\exp(-x))$$
$$v(x, y) = \sqrt{2}\exp(x/2)\sin(y\exp(-x))$$

Then

$$F(0, y + 2k\pi) = F(0, y)$$

so F is not injective even though

$$\det F'(x) = 1.$$

JC is True for Quadratic Maps in all Dimensions

Quadratic maps $F : \mathbb{C}^n \to \mathbb{C}^n$ satisfy

The Mean-Value Formula

$$F(x) - F(y) = F'\left(\frac{x+y}{2}\right)(x-y).$$

Consequently,

$$\det F'(x) \equiv \text{constant} \neq 0$$

implies that $F(x)$ is *injective*.

Polyflow Formulations of Keller's Jacobian Conjecture

Let k denote $\mathbb{R}$ or $\mathbb{C}$.
Let F be a normalized Keller map of k^n into itself: $F(x)$ is polynomial, $F(0) = 0$ & $\det F'(x) \equiv 1$.

Then injectivity of F is equivalent to each:

1986: $\forall a \in k^n$, the vector field $F'(x)^{-1}a$ has a polyflow.
[Meisters & Olech]

1995: The vector field $-[F'(x)]^{-1}F(x)$ has a polyflow.
[Coomes]

Derivation Formulations of Keller's Jacobian Conjecture

Call a derivation D on the polynomial ring

$$\mathbb{C}^{[n]} := \mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n]$$

- **locally nilpotent** if $\forall g \in \mathbb{C}^{[n]}\ \exists m\ D^m g = 0$.
- **locally finite** if $\forall g \in \mathbb{C}^{[n]}$ $\text{dimSpan}\{D^\alpha g\} < \infty$.

The **torsion space** $T(D)$ of D is defined as the set:

$$\{a \in \mathbb{C}^{[n]} \,|\, q(D)a = 0 \text{ for some } q(D) \in \mathbb{C}[D]\,\}.$$

Let F be a normalized Keller map of $\mathbb{C}^n$ into itself: i.e., $F(x)$ is polynomial, $F(0) = 0$ & $\det F'(x) \equiv 1$.

Let D be the derivation $-F'(x)^{-1}F(x) \cdot \nabla$.

Let d_i be the derivation $((F'(x))^{-1}\mathbf{e}_i)^T\nabla$, $1 \le i \le n$.

Then injectivity of F is equivalent to each:

1991: $T(D) = \mathbb{C}^n$. [Coomes & Zurkowski]

1992: All derivations d_i are locally nilpotent. [v. d. Essen]

1992: All derivations d_i are locally finite.
[van den Essen via Nousiainen & Sweedler 1983]

1995: $T(D)$ is algebraically closed in $\mathbb{C}[x]$. [Coomes]

A New Approach to Keller's Jacobian Conjecture

Poincaré. For $f : \mathbb{C}^n \to \mathbb{C}^n$ analytic at 0, $f(0) = 0$, and eigenvalues $(s_1, \ldots, s_n)$ of $f'(0)$ **not resonant**, there is a formal power series $h(x)$ so that *locally* at 0

$$h \circ f \circ h^{-1}(x) = f'(0)x.$$

Deng, Meisters & Zampieri (March 1994).

So perhaps, for some complex s, there is an h_s so that

$$h_s \circ sF_A \circ h_s^{-1}(x) = sx$$

even *globally* for Drużkowski's **cubic-linear** maps

$$F_A(x) := x - [\text{diag}(Ax)]^2 Ax,$$

for $n \times n$ matrix A, if $\det F'_A(x) \equiv 1$.

Dfn: Call matrix A **admissible** if $\det F'_A(x) \equiv 1$.

Dfn: Call matrix A **good** if F_A is injective.

ABSTRACT

The 1939 Jacobian Conjecture of O.-H. Keller remains open, as well as numerous related questions, but the quest for its solution has helped to completely solve, in all dimensions, the 1960 Conjecture of Larry Markus & Hidehiko Yamabe on Global Asymptotic Stability of a rest point of a nonlinear, class $\mathcal{C}^1$, autonomous system of differential equations.

Hidehiko Yamabe 1923–1960
Japanese-American Mathematician

Yamabe was born August 22, 1923, in Ashiya, in the Hyogoken prefecture, near Osaka, Japan. Fifth son of Takehiko and Rei Yamabe. Loved sports, but his health problem prevented him from playing much. He knew, as he told friends later, that his parents had been warned by a doctor not to expect him to live much more than 20 years. Spent two and a half years at The Third Senior High School in Kyoto 1942–44; Student of S. Iyanaga at Tokyo University; Master of Science (Rigakushi) 1947. Then appointed assistant at Osaka University and held that position until 1952. The years at Kyoto, Tokyo & Osaka were rough times due to World War II. Hidehiko Married Etsuko in 1952; came to the Institute for Advanced Study, Princeton, as an assistant to Dean Montgomery. There he completed his famous solution to Hilbert's Fifth Problem; 1^{st} child Kimiko was born; received Doctor's Degree (Rigaku Hakushi) April, 1954, from Osaka University. Assistant Professor at the University of Minnesota 1954–57; 2^{nd} child Noriko 1956; Associate Professor 1957–58. Returned 1958–59 to Osaka as Professor to help the younger generation, but came back to the University of Minnesota as Associate Professor for AY 59–60. Then, fall of 1960, to Northwestern University, Evanston, Illinois, as Professor. One evening he entered an Evanston Hospital with a severe headache and, after five days of unconsciousness, died November 20, 1960, of a subarachnoid hemorrhage.

In 1953 Yamabe obtained the final answer to **Hilbert's Fifth Problem**: *A connected locally compact group G is a projective limit of a sequence of Lie groups; and, if G has no small subgroups, then it is a Lie group.*

References:

1. G. Morikuni, **Hidehiko Yamabe [1923–1960]**, *Osaka Math. J.* **13** (1961).
2. R. P. Boas, **Collected Works of Hidehiko Yamabe**, Notes on Mathematics and its Applications, Editors: J.T. Schwartz (Courant Institute) & M. Lévy (Université de Paris), *Gordon and Breach* 1967; 150 Fifth Avenue, New York, N. Y. 10011.
3. D. Montgomery & L. Zippin, **Topological Transformation Groups**, *Interscience* 1955; Yamabe's 1953 Theorem on Lie Groups is on page 107.

SOME PROBLEMS STILL OPEN

For polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$

with $F(0) = 0$ **and** $\det F^{'}(x) \equiv 1$**;**

esp. if $F(x) = x - H(x)$ **with** $H(tx) = t^3 H(x)$**;**

or $F_A(x) = x - H_A(x)$ **&** $H_A(x) = [\mathbf{diag}(Ax)]^2 Ax$**.**

---

1. **Keller's JC:** Is F *injective*?

2. Is the solution of $\frac{dx}{dt} = F^{'}(x)^{-1} v$ a *polyflow* $\forall v$?

3. Is the solution of $\frac{dx}{dt} = -[F^{'}(x)]^{-1} F(x)$ a *polyflow*?

4. $\exists$? *injective* h_s so that $h_s \circ sF \circ h_s^{-1}(x) \equiv sx$?

5. Can $\frac{dx}{dt} = c - x + H(x)$ have *two rest points*?

6. **The Phantom Eigenvector Problem:**
   Can $3[\mathrm{diag}(Ax)][\mathrm{diag} Ay)]A(x - y) = (x - y)$
   for some $x \neq y$ if $[\mathrm{diag} Ax)]^2 A$ is *nilpotent* $\forall\, x$?