# POLYOMORPHISMS CONJUGATE TO DILATIONS

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## 1. Background and terminology

Consider **polynomial** maps $f : \mathbf{C}^n \to \mathbf{C}^n$ and their **dilations** $sf(x)$ by complex scalars $s$. That is, maps $f$ whose components $f_i$ are polynomials with complex coefficients in the $n$ variables $(x_1, x_2, \ldots, x_n) = x \in \mathbf{C}^n$. The question, first raised by O.-H. Keller in 1939 [10] for polynomials over the integers but now also raised for complex polynomials and, as such, known as **The Jacobian Conjecture** (**JC**), asks whether a *polynomial* map $f$ with nonzero constant Jacobian determinant $\det f'(x)$ need be a **polyomorphism**: I.e., *bijective* with *polynomial inverse*. It suffices to prove *injectivity* because in 1960–62 it was proved, first in dimension 2 by Newman [19] and then in all dimensions by Białynicki-Birula and Rosenlicht [4], that, for polynomial maps, surjectivity follows from injectivity; and furthermore, under Keller's hypothesis, the inverse $f^{-1}(x)$ will be polynomial, at least in the complex case, if the polynomial map is bijective. The group of all polyomorphisms of $\mathbf{C}^n$ is denoted $\mathbf{GA}_n(\mathbf{C})$. It is isomorphic to the group $\mathbf{Aut}\,\mathbf{C}[\mathbf{x}]$ of automorphisms $\sigma$ of the polynomial ring $\mathbf{C}[\mathbf{x}]$ by means of the correspondence $\phi(f) = \sigma$ where $\sigma(x_i) = f_i(x)$. Polynomial maps $f(x)$ satisfying $\det f'(x) = const \neq 0$ are called **Keller maps**. We can and do assume that $f(0) = 0$ and $f'(0) = I$. Five main problems arise:

**Problem#1**. Classify all **Keller maps** $f : \mathbf{C}^n \to \mathbf{C}^n$. Open for $n \geq 2$.

**Problem#2**. Classify all **polyomorphisms** $f : \mathbf{C}^n \to \mathbf{C}^n$. See [3, 7, 17].

The Theorem of van der Kulk & Jung [7, 9, 11, 17] which states that *every polyomorphism of* $\mathbf{C}^2$ *is the composition of a finite number of affine and triangular ones* solves Problem#2 for $n = 2$. Thus the structure of $\mathbf{GA}_n(\mathbf{C})$ is well understood for $n = 2$; but it's an open problem for $n \geq 3$.

**Problem#3**. Is every Keller map a polyomorphism? (This is Keller's **JC**.)

It has been shown [2] that to prove **JC** $\forall n \geq 2$ it suffices to prove it (but also $\forall n \geq 2$) for maps of the form $f(x) = x - g(x)$ where $g(tx) \equiv t^3 g(x)$. Then Keller's condition $\det f'(x) = const \neq 0$ is equivalent to nilpotence of $g'(x)$; for a proof see [12, Lemma 1(c) page 112]. **JC** is open for $n \geq 2$.

**Problem#4**. Classify all **cubic-homogeneous** $g(x)$ satisfying $g'(x)^n \equiv 0$. Open for $n \geq 5$. See Hubbers [8]. Drużkowski [6] reduced **JC** to the case of **cubic-linear** maps $f(x) = x - [diag(Ax)]^2 Ax$. We say the **kernel-matrix** $A$ is **admissible** if $g'_A(x) \equiv 3[diag(Ax)]^2 A$ is nilpotent for all $x$. Matrices $A$ and $D$ are called **cubic-similar** [13] (denoted $A \overset{cubic}{\sim} D$) if, for some matrix $P$ in $\mathbf{GL}_n(\mathbf{C})$, $g'_A(Pu)P = Pg'_D(u)$, $\forall u \in \mathbf{C}^n$. The rank of $A$ and the nilpotence-indices of $g'(x)$ and $[diag(Ax)][diag(Ay)]A$ (but not that of $A$) are cubic-similarity invariants.

**Problem#5**. Classify all admissible matrices $A$. For $n \leq 4$ see [8, 13, 14]. Open for $n \geq 5$. *Quadratic* analogues of Problems #4 & #5 are also open.

## 2. What's in this paper & Proposition 1

In this paper we use examples of polyomorphisms $f$, many taken from [12, 13, 14, 15, 16], to construct various examples of *global conjugations* $h_s(x) \equiv h(s, x)$ of $s\, f(x)$ to $L_s(x) \equiv s\, x$. All of the examples of $h_s$ that we worked out (except for the *nonhomogeneous* one discussed in §§ 6 & 6.1) turned out to be *1-parameter families of polyomorphisms*. The degrees of our examples are tabulated in §§ 3 & 6. Five of the examples themselves are listed in §§ 5, 6, & 7. Three more examples are given in the Appendix. What we have been able to *prove* globally is stated below in Proposition 1. Our attempts [5] to adapt the Poincaré-Siegel Theory & Methods (cf. §§ 9 & A.1) directly to our case led mostly only to *local* results.—Reality or just us?

**Proposition 1** *Let* $f : \mathbf{C}^n \to \mathbf{C}^n$ *be a nonlinear polynomial mapping of the form* $f(x) = x - g(x)$ *with* $g(0) = 0$, $g'(0) = 0$, *and* $\det f'(x) = 1$ *for all* $x$ *in* $\mathbf{C}^n$. *Then, evidently, for each complex number* $s$, *except for certain roots of unity, there is a (unique) formal power series* $h_s(x) \equiv h(s, x)$ *such that* $\det h'_s(0) \neq 0$ *and*

$$h(s, s\, f(x)) = s\, h(s, x) \quad for\ all\ x \in \mathbf{C}^n. \tag{1}$$

*Furthermore:*

1. $h_s(0) = 0$*; and we may assume without loss of generality* $h'_s(0) = I$.
2. $h_0(x) = x - g(x) = f(x)$. *That is,* $\lim_{s\to 0} h_s(x) = x - g(x) = f(x)$.
3. *If* $f$ *is injective, then, for* $k_s(x) \overset{def}{=} s\, h\left(\frac{1}{s}, \frac{x}{s}\right) \equiv L_s \circ h_{1/s} \circ L_s^{-1}(x)$,
   (a) $k(s, s\, f^{-1}(x)) = s\, k(s, x)$ *(i.e.,* $k_s$ *is to* $f^{-1}$ *as* $h_s$ *is to* $f$*), and*
   (b) $f^{-1}(x) = k_0(x) = \lim_{s\to 0} s\, h\left(\frac{1}{s}, \frac{x}{s}\right) = \lim_{t\to\infty} \frac{1}{t} h(t, tx)$.

*4. The map $f$ is injective iff for each $x$, $\lim_{s\to\infty} h_s(x) = x$.*
*5. For maps $f(x) = x - g(x)$ with $g(tx) = t^2 g(x)$, we find*

$$h_s(x) = x + \frac{g(x)}{(s-1)} + \frac{sg'(x)g(x)}{(s-1)(s^2-1)} + \frac{s(1+s^2)g(g(x))}{(s-1)(s^2-1)(s^3-1)} + \cdots \quad (2)$$

*6. For maps $f(x) = x - g(x)$ with $g(tx) = t^3 g(x)$, we find*

$$h_s(x) = x + \sum_{k=1} c_k(s) H_k(x) \quad (3)$$

*where $H_k(tx) = t^{2k+1} H_k(x)$ and $c_k(s) = p_k(s)/(s^2-1)\cdots(s^{2k}-1)$.*

TABLE 1. Polynomials $p_k(s)$ and vector-functions $H_k(x)$ in (3)

| $k \ge 1$ | $p_k(s)$ | $H_k(x)$ | $\deg H_k$ |
|---|---|---|---|
| 1 | 1 | $g(x)$ | 3 |
| 2 | $s^2$ | $g'(x)g(x)$ | 5 |
| 3 | $s^2(1+s^4)$ | $g'(g(x))x$ | 7 |
| 4 | $s^2(1+s^2)(1-s^2+3s^4-s^6+s^8)$ | $g(g(x))$ | 9 |

For the 5D cubic-homogeneous, non-cubic-linear, examples $H02$–$H07$ in Table 3 we find a *different* term of degree 7 in the series (3) for $h_s(x)$: Instead of $g'(g(x))x$ as listed above for cubic-linear maps, we find $H_3(x) = g'(x)^2 g(x)$ and $p_3(s) = s^6$. But not so for Rusek's example after it is reduced to a cubic-homogeneous!

## 3. Proof of Proposition 1

*Proof of 1.* If there is a mapping $h_s$ satisfying Eq. (1), then at $x = 0$ we find $h_s(0) = 0$ (because $s \neq 1$). Since, for each linear map $L$, $h_s$ can be replaced by $L \circ h_s$ in Eq. (1), we may assume without loss of generality that $h_s'(0) = I$; just take $L = h_s'(0)^{-1}$. □

*Proof of 2.* Differentiating Eq. (1) with respect to $s$ we get

$$\frac{\partial h}{\partial s}(s, s\,f(x)) + \frac{\partial h}{\partial x}(s, s\,f(x))f(x) = h(s,x) + s\frac{\partial h}{\partial s}(s,x).$$

At $s = 0$ this yields

$$\frac{\partial h}{\partial s}(0,0) + \frac{\partial h}{\partial x}(0,0)f(x) = h(0,x). \quad (4)$$

But since $\frac{\partial h}{\partial x}(s,0) = h'_s(0) = I$ for all s, $\frac{\partial h}{\partial x}(0,0) = I$ too. Thus (4) becomes

$$\frac{\partial h}{\partial s}(0,0) + f(x) = h(0,x).$$

But $f(0) = h(s,0) = 0$, so $\frac{\partial h}{\partial s}(0,0) = 0$ and $f(x) = h(0,x)$ as claimed. □

*Proof of 3(a).* By replacing $s$ by $1/s$ in Eq. (1) we obtain

$$h_{1/s}(f(x)/s) = h_{1/s}(x)/s.$$

Then replacing $x$ by $f^{-1}(x)$, we obtain

$$h_{1/s}(x/s) = h_{1/s}(f^{-1}(x))/s.$$

Exchanging sides and multiplying through by $s^2$ we obtain (as desired):

$$k(s, s\,f^{-1}(x)) = s\,h_{1/s}((s\,f^{-1}(x))/s) = s\,(s\,h_{1/s}(x/s)) = s\,k(s,x).$$

*Proof of 3(b).* Follows from *3*(a) and part *2.* above. □

*Proof of 4.* First assume $h_\infty(x) = x$: Then if $f(x_1) = f(x_2)$ it follows from (1) that $h_s(x_1) = h_s(x_2)$ for all but finitely many complex numbers $s$. But as $s$ tends to $\infty$, $h_s(x)$ tends to $x$ for every $x \in \mathbf{C}^n$. Consequently, $x_1 = x_2$, so that $f$ is injective. Next assume that $f^{-1}$ exists. Then

$$x = f^{-1}(f(x)) = \lim_{s\to 0} s\,h\left(\frac{1}{s}, \frac{f(x)}{s}\right) = \lim_{t\to\infty} \frac{1}{t} h(t, t\,f(x)) = \lim_{t\to\infty} h(t,x).$$

□

*Proof of 5.* Under the assumption that $g$ is quadratic-homogeneous, repeated partial differentiation of Eq. (1) with respect to the components of $x$ leads (uniquely) to the Formula (2). That it is valid (as far as it goes) is verified by the examples of $h_s$ for quadratic-homogeneous maps $f$ that we worked out by using Formula (2). These examples of $h_s$ were first computed by using Formula (2) and then verified independently of Formula (2) by checking directly that they satisfy Eq. (1). □

*Proof of 6.* Under the assumption that $g$ is cubic-homogeneous, repeated partial differentiation of Eq. (1) leads to the Formula (3). That it is valid (as far as it goes) was verified by examples of $h_s$ for cubic-homogeneous maps $f$ that we worked out by using Formula (3) and the entries in the Table 1. These examples of $h_s$ were then verified independently by checking directly that they satisfy Eq. (1). □

This completes the proof of Proposition 1. **Q.E.D.**

TABLE 2. Degrees of Cubic-Linear Examples

| Dim & Name $A$ | $\deg f$ | $\deg f^{-1}$ | $\deg h_s$ | $\deg h_s^{-1}$ | The Map |
|---|---|---|---|---|---|
| 2D-$J(1.2)^T$ | 3 | 3 | 3 | 3 | Not given |
| 2D-J(1.2) | 3 | 3 | 3 | 3 | See Sec. 5 |
| 3D-J(1.2) | 3 | 3 | 3 | 3 | Not given |
| 3D-J(2.3) | 3 | 9 | 9 | 9 | See Sec. 5 |
| 4D-J(1.2) | 3 | 3 | 3 | 3 | Not given |
| 4D-J(2.2) | 3 | 3 | 3 | 3 | Not given |
| 4D-J(2.3) | 3 | 9 | 9 | 9 | Not given |
| 4D-N(2.3) | 3 | 9 | 9 | 9 | Not given |
| 4D-J(3.4) | 3 | 27 | 27? | Incomplete | Not given |
| 4D-N(3.4) | 3 | 27 | 27? | ditto | Not given |
| 5D-J(1.2) | 3 | 3 | 3 | 3 | Not given |
| 5D-J(2.2) | 3 | 3 | 3 | 3 | Not given |
| 5D-J(2.3) | 3 | 9 | 9 | 9 | Not given |
| 5D-J(3.3) | 3 | 9 | 9 | 9 | Not given |
| 5D-J(3.4) | 3 | 27 | 27? | Incomplete | Not given |
| 5D-J(4.5) | 3 | 81 | 81? | ditto | Not given |
| 5D-N(2.3a) | 3 | 9 | 9 | 9 | Not given |
| 5D-N(3.3a) | 3 | 9 | 9 | 9 | Not given |
| 5D-N(3.4a) | 3 | 27 | 27? | Incomplete | Not given |
| 5D-N(3.4b) | 3 | 27 | 27? | ditto | Not given |
| 5D-N(4.5a) | 3 | 81 | 81? | ditto | Not given |

Table 2 compares the degrees of $f$, $f^{-1}$, $h_s$, and $h_s^{-1}$ satisfying Eq. (1) for 21 cubic-linear polyomorphisms $f(x) = x - [diag(Ax)]^2 Ax$ defined by a kernel matrix $A$ ($= J(1.2)^T, J(1.2), \ldots$). These matrices $A$ are 20 of the 27 admissible matrices listed on the next page. They were given in [13, 14] as **distinct representatives** of **cubic-similarity equivalence classes**. In each case, the map $h_s(x)$ defined by Eq. (1), turns out to be a *polyomorphic-conjugation* of $sf(x)$ to $sx$. In each of the names $X(\rho.\nu)$ in the left column, $\rho$ is the **rank** of $A$ and $\nu$ is the **nilpotence-index** of $g'_A(x) = 3[diag(Ax)]^2 A$. The name $X = J$ means it is one of the usual Jordan normal forms; but the name $X = N$ means it is not a Jordan normal form. Note that for each *triangularizable* cubic-linear polyomorphism $f(x)$ in this table

$$deg(h_s^{-1}) = deg(h_s) = max\{deg\, f\,,\, deg\, f^{-1}\} = \Big(min\{deg\, f\,,\, deg\, f^{-1}\}\Big)^{\nu-1} = 3^{\nu-1}. \quad (5)$$

All entries are in agreement with Rusek [22, Conjecture 5.5, page 20].

## 4. Kernel matrices $A$ for cubic-linear maps

Here are the (mutually inequivalent) kernel matrices $A$ used in Table 2: Representatives for cubic-similarity equivalence classes discussed in [13, 14].

**Dimensions** $N = 2$ **&** $N = 3$:

$$J(1.2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad J(1.2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad J(2.3) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

**Dimension** $N = 4$:

$$\underset{J(1.2)}{\begin{bmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}} \quad \underset{J(2.3)}{\begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}} \quad \underset{J(3.4)}{\begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}}$$

$$\underset{J(2.2)}{\begin{bmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}} \quad \underset{N(2.3)}{\begin{bmatrix} 0&0&1&0 \\ 0&0&1&1 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}} \quad \underset{N(3.4)}{\begin{bmatrix} 0&1&1&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}}$$

**Dimension** $N = 5$:

$$\underset{J(1.2)}{\begin{bmatrix} 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}} \quad \underset{J(2.2)}{\begin{bmatrix} 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}} \quad \underset{J(2.3)}{\begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}} \quad \underset{J(3.3)}{\begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}}$$

$$\underset{J(3.4)}{\begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}} \quad \underset{J(4.5)}{\begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}} \quad \underset{N(2.3a)}{\begin{bmatrix} 0&0&1&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}} \quad \underset{N(3.3a)}{\begin{bmatrix} 0&0&1&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}}$$

$$\underset{N(3.4a)}{\begin{bmatrix} 0&1&1&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}} \quad \underset{N(3.4b)}{\begin{bmatrix} 0&0&1&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}} \quad \underset{N(4.5a)}{\begin{bmatrix} 0&1&1&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}} \quad \underset{N(4.5b)}{\begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}}$$

$$\underset{N(3.4c)}{\begin{bmatrix} 0&0&1&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&1 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}} \quad \underset{N(3.4d)}{\begin{bmatrix} 0&1&0&1&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}} \quad \underset{N(4.5c)}{\begin{bmatrix} 0&1&1&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}} \quad \underset{N(4.5d)}{\begin{bmatrix} 0&1&0&1&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}}$$

$$\underset{P(4.5c)}{\begin{bmatrix} 0&1&1&0&0 \\ 0&0&1&a&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}} \quad \underset{P(4.5c2)}{\begin{bmatrix} 0&1&1&a&0 \\ 0&0&1&b&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}}$$

This is not a complete list of mutually inequivalent 5D-representatives.

## 5. Two cubic-linear maps showing $f$, $f^{-1}$, $h_s$ and $h_s^{-1}$

$$y = f(x) = x - g(x) = x - [diag(Ax)]^2 Ax$$

**2D-J(1.2)**
Degrees $\{f,\, f^{-1},\, h_s,\, h_s^{-1}\} = \{3, 3, 3, 3\}$, $g'(x)^2 = 0$, $B(x,y)^2 = 0$, and $g(g(x)) = 0$.

$$f(x) = \begin{bmatrix} x_1 - {x_2}^3 \\ x_2 \end{bmatrix} \qquad f^{-1}(y) = \begin{bmatrix} y_1 + {y_2}^3 \\ y_2 \end{bmatrix}$$

$$h_s(x) = \begin{bmatrix} x_1 + \frac{{x_2}^3}{s^2-1} \\ x_2 \end{bmatrix} \qquad h_s^{-1}(y) = \begin{bmatrix} y_1 - \frac{{y_2}^3}{s^2-1} \\ y_2 \end{bmatrix}$$

**3D-J(2.3)**
Degrees $\{f,\, f^{-1},\, h_s,\, h_s^{-1}\} = \{3, 9, 9, 9\}$, $g'(x)^3 = 0$, $B(x,y)^3 = 0$, and $g(g(g(x))) = 0$.

$$f(x) = \begin{bmatrix} x_1 - {x_2}^3 \\ x_2 - {x_3}^3 \\ x_3 \end{bmatrix} \qquad f^{-1}(y) = \begin{bmatrix} y_1 + {y_2}^3 + 3\,{y_2}^2\,{y_3}^3 + 3\,y_2\,{y_3}^6 + {y_3}^9 \\ y_2 + {y_3}^3 \\ y_3 \end{bmatrix}$$

$$h_s(x) = \begin{bmatrix} \left\{ \begin{array}{c} x_1 + \frac{{x_2}^3}{s^2-1} + \frac{3\,s^2\,{x_2}^2\,{x_3}^3}{(s-1)^2(s+1)^2(s^2+1)} + \\ \frac{3\,s^2\left(s^4+1\right)x_2\,{x_3}^6}{(s-1)^3(s+1)^3(s^2+1)(s^2-s+1)(s^2+s+1)} + \\ \frac{s^2\left(1-s^2+3s^4-s^6+s^8\right){x_3}^9}{(s-1)^4(s+1)^4(s^2+1)(s^2-s+1)(s^2+s+1)(s^4+1)} \end{array} \right\} \\ x_2 + \frac{{x_3}^3}{s^2-1} \\ x_3 \end{bmatrix}$$

$$h_s^{-1}(y) = \begin{bmatrix} \left\{ \begin{array}{c} y_1 + \frac{{y_2}^3}{s^2-1} + \frac{3{y_2}^2{y_3}^3}{(s-1)^2(s+1)^2(s^2+1)} - \\ \frac{3y_2{y_3}^6}{(s^2-1)^3(s^4+s^2+1)} + \frac{{y_3}^9}{(s^2-1)^4(s^6+s^4+s^2+1)} \end{array} \right\} \\ y_2 - \frac{{y_3}^3}{s^2-1} \\ y_3 \end{bmatrix}$$

## 6. Rusek's Example & Other Non-cubic-linear Examples

The 2D-example represented by the first line in Table 3 is qualitatively different from all the other examples discussed in this paper: It is a Keller map of the form $f(\vec{x}) = \vec{x} - g(\vec{x})$ with $g(\vec{x})$ *not homogeneous*. Here it is with $\vec{x} = (x, y)$: $f(x, y) = (x + (y + x^2)^2, y + x^2)$. It is the composition $f = (L \circ QL) \circ (L \circ QL)$, where $L(x, y) = (y, x)$ is linear and $QL(x, y) = (x, y + x^2)$ is quadratic-linear. The series for its Schröder function is $h(s, x, y) =$

$$\begin{bmatrix} x - \frac{y^2}{(s-1)} + \frac{2x^2y}{(s-1)^2(s+1)} - \frac{4s^2xy^3}{(s-1)^3(s+1)(s^2+s+1)} - \frac{(s^2+1)x^4}{(s-1)^3(s+1)(s^2+s+1)} \\ y - \frac{x^2}{(s-1)} + \frac{2sxy^2}{(s-1)^2(s+1)} - \frac{4sx^3y}{(s-1)^3(s+1)(s^2+s+1)} - \frac{s(s^2+1)y^4}{(s-1)^3(s+1)(s^2+s+1)} \end{bmatrix} + \cdots$$

By Eq. (5) $\deg h_s$ should be 4. But there are terms of degree 5 and higher:

$$+ \begin{bmatrix} \frac{4s^2(s^3+s+3)x^3y^2}{(s-1)^4(s+1)^2(s^2+1)(s^2+s+1)} + \frac{2s^2(s^2-s+1)y^5}{(s-1)^4(s+1)(s^2+1)(s^2+s+1)} \\ \frac{4s(3s^3+s^2+1)x^2y^3}{(s-1)^4(s+1)^2(s^2+1)(s^2+s+1)} + \frac{2s(s^2-s+1)x^5}{(s-1)^4(s+1)(s^2+1)(s^2+s+1)} \end{bmatrix} + \cdots$$

Nevertheless, the Formula *3*(b) of Prop. 1 gets $f^{-1}$ even if $h_s(x)$ is an infinite series: Using all terms of degree $\leq 7$ (degrees 6 & 7 are given in § 6.1 below)

$$f^{-1}(x, y) = \lim_{t \to \infty} \frac{1}{t} h(t, tx, ty) = \left(x - y^2, y - (x - y^2)^2\right).$$

Indeed, all terms of $h(t, tx, ty)/t$ not part of $f^{-1}$ become zero as $t \to \infty$. However, it is shown in § 6.2 that the Schröder function $h(s, x)$ corresponding to the BCW-reduction of Rusek's example to cubic-homogeneous form is again a *polyomorphism* for each $s$.

### 6.1. DEGREE 6 & 7 SCHRÖDER-TERMS FOR RUSEK'S 2D

Here are a few more terms of the series for the 1-parameter family of conjugations $h_s$ which corresponds to Rusek's non-homogeneous polynomial map $f(x, y) = (x + (y + x^2)^2, y + x^2)$ discussed above.

**All terms of $h_s(x)$ of degree 6:**

$$+ \begin{bmatrix} \frac{-2s^2(5+3s+8s^3+18s^4+3s^5+3s^7)\,x^2y^4}{(s-1)^5(s+1)^2(s^2+1)(s^2+s+1)(s^4+s^3+s^2+s+1)} \\ \\ \frac{-2s(3+3s^2+18s^3+8s^4+3s^6+5s^7)\,x^4y^2}{(s-1)^5(s+1)^2(s^2+1)(s^2+s+1)(s^4+s^3+s^2+s+1)} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{-4s^2(3+2s+2s^3+3s^4)\,x^5y}{(s-1)^5(s+1)^2(s^2+1)(s^2+s+1)(s^4+s^3+s^2+s+1)} \\ \\ \frac{-4s^4(3+2s+2s^3+3s^4)\,xy^5}{(s-1)^5(s+1)^2(s^2+1)(s^2+s+1)(s^4+s^3+s^2+s+1)} \end{bmatrix} + \cdots$$

**All terms of $h_s(x)$ of degree 7:**

$$+\begin{bmatrix} \frac{4s^3(1-s+2s^2+7s^3+2s^5+7s^7+2s^8-s^9+s^{10})\,xy^6}{(s-1)^6(s+1)^2(s^2+1)(s^2-s+1)(s^2+s+1)^2(s^4+s^3+s^2+s+1)} \\ \\ \frac{4s(1-s+2s^2+7s^3+2s^5+7s^7+2s^8-s^9+s^{10})x^6y}{(s-1)^6(s+1)^2(s^2+1)(s^2-s+1)(s^2+s+1)^2(s^4+s^3+s^2+s+1)} \end{bmatrix}$$

$$+\begin{bmatrix} \frac{4s^2(5+6s+11s^3+36s^4+26s^5+10s^6+6s^7+15s^8+15s^9)\,x^4y^3}{(s-1)^6(s+1)^3(s^2+1)(s^2-s+1)(s^2+s+1)^2(s^4+s^3+s^2+s+1)} \\ \\ \frac{4s^4(15+15s+6s^2+10s^3+26s^4+36s^5+11s^6+6s^8+5s^9)x^3y^4}{(s-1)^6(s+1)^3(s^2+1)(s^2-s+1)(s^2+s+1)^2(s^4+s^3+s^2+s+1)} \end{bmatrix}$$

$$+\begin{bmatrix} \frac{4s^2(1+s^3+s^4+s^5+s^8)\,x^7}{(s-1)^6(s+1)^2(s^2+1)(s^2-s+1)(s^2+s+1)^2(s^4+s^3+s^2+s+1)} \\ \\ \frac{4s^4(1+s^3+s^4+s^5+s^8)\,y^7}{(s-1)^6(s+1)^2(s^2+1)(s^2-s+1)(s^2+s+1)^2(s^4+s^3+s^2+s+1)} \end{bmatrix} + \cdots$$

The **denominator zeros** $\pm 1$, $\pm i$, $\pm\frac{1}{2}\pm\frac{\sqrt{3}\,i}{2}$, $\frac{-1\pm\sqrt{5}\pm\sqrt{2}\,\sqrt{-5\mp\sqrt{5}}}{4}$ are all on the unit circle. Indeed, they are roots of unity.

If in Eq. (1) we use only a truncated form $\widetilde{h}\,(s,x,y)$ of $h(s,x,y)$, say all terms only up to a certain degree, then we find that all terms in Eq. (1) having the same degree as those present in $\widetilde{h}\,(s,x,y)$ are annihilated. Thus, e.g., if $\widetilde{h}\,(s,x,y)$ contains only the terms of degrees 1 though 7 (as written above), then the difference $\widetilde{h}(s,sf(x)) - s\widetilde{h}(s,x)$ contains only terms of degrees greater than 7.

## 6.2. REDUCTION OF 2D-RUSEK TO 5D-CUBIC-HOMOGENEOUS

Now we show that the Schröder function $h(s,x)$ corresponding to the cubic-homogeneous polyomorphism $F(x)$, obtained by applying the Bass-Connell-Wright (BCW) Reduction-of-Degree to Rusek's nonhomogeneous polyomorphism [21] $f(x,y) = (x+(y+x^2)^2, y+x^2)$, is itself a *polyomorphism* for each $s$. At Curaçao David Wright helped me get the reduction

$$F(x) = \begin{bmatrix} x_1 + x_5({x_2}^2 - {x_3}^2 - x_4x_5) \\ x_2 + {x_1}^2x_5 \\ x_3 + {x_1}^2x_5 \\ x_4 + 2{x_1}^2(x_2 - x_3) \\ x_5 \end{bmatrix}.$$

We easily find

$$F^{-1}(y) = \begin{bmatrix} y_1 - {y_2}^2 y_5 + {y_3}^2 y_5 + y_4 {y_5}^2 \\ y_2 - y_5(y_1 - {y_2}^2 y_5 + {y_3}^2 y_5 + y_4 {y_5}^2)^2 \\ y_3 - y_5(y_1 - {y_2}^2 y_5 + {y_3}^2 y_5 + y_4 {y_5}^2)^2 \\ y_4 + 2(y_3 - y_2)(y_1 - {y_2}^2 y_5 + {y_3}^2 y_5 + y_4 {y_5}^2)^2 \\ y_5 \end{bmatrix}$$

It has degree 7 so $h(s,x)$ must have $x$-degree *at least* 7. In order to obtain the Schröder function $h(s,x)$ for this 5-dimensional polyomorphism $F(x)$ I used the general formula which follows from Eq. (1) for this particular type of polynomial automorphism: Namely,—

$$h(s,x) = x + g(x)/(s^2-1) + s^2 Dg(x)g(x)/((s^2-1)(s^4-1)) + \\ s^2(s^4+1)Dg(g(x))x/((s^2-1)(s^4-1)(s^6-1)).$$

So for this particular example we obtain the explicit formula $h(s,x) =$

$$\begin{bmatrix} x_1 + \frac{x_5(-{x_2}^2+{x_3}^2+x_4x_5)}{s^2-1} \\ x_2 - \frac{{x_1}^2 x_5}{s^2-1} - \frac{2s^2 x_1 {x_5}^2(-{x_2}^2+{x_3}^2+x_4x_5)}{s^6-s^4-s^2+1} - \frac{s^2(s^4+1){x_5}^3(-{x_2}^2+{x_3}^2+x_4x_5)^2}{s^{12}-s^{10}-s^8+s^4+s^2-1} \\ x_3 - \frac{{x_1}^2 x_5}{s^2-1} - \frac{2s^2 x_1 {x_5}^2(-{x_2}^2+{x_3}^2+x_4x_5)}{s^6-s^4-s^2+1} - \frac{s^2(s^4+1){x_5}^3(-{x_2}^2+{x_3}^2+x_4x_5)^2}{s^{12}-s^{10}-s^8+s^4+s^2-1} \\ x_4 + \frac{2{x_1}^2(x_3-x_2)}{(s^2-1)} + \frac{4s^2 x_1(x_3-x_2)x_5(-{x_2}^2+{x_3}^2+x_4x_5)}{s^6-s^4-s^2+1} + \frac{2s^2(s^4+1)(x_3-x_2){x_5}^2(-{x_2}^2+{x_3}^2+x_4x_5)^2}{s^{12}-s^{10}-s^8+s^4+s^2-1} \\ x_5 \end{bmatrix}$$

Note that it is indeed a *polyomorphism* (of degree 7). Thus, although the Schröder function $h(s,x)$ for the *nonhomogeneous* polyomorphism $f(x,y) = \big(x + (y+x^2)^2, y + x^2\big)$ is not itself polynomial in $x$, the Schröder function for its reduction to *"cubic-homogeneous"* form *is* a polyomorphism .

Table 3 compares the degrees of $f$, $f^{-1}$, $h_s$, and $h_s^{-1}$ as defined by Eq. (1), for some *non-cubic-linear* maps. All but the first entry are of the form $f(x) = x - g(x)$ with $g(tx) = t^3 g(x)$ and $g'(x)^n = 0$. It shows that part of Eq. (5) persists for non-cubic-linear maps of the form $x$+homogeneous:

$$deg\, h_s^{-1} = deg\, h_s = max\{deg\, f, deg\, f^{-1}\} \leq 3^{\nu-1}.$$

In the names 5D-Hab($\rho.\nu$) in the left column of Table 3, $\rho$ is the rank and $\nu$ is the *nilpotence-index* of $g'_A(x)$. The Jacobian $g'(x)$ is called **Strongly Nilpotent (S.N.)** if arbitrary products $g'(x)g'(y) \overset{n}{\cdots} g'(z)$ are zero for all $x, y, \ldots, z$ in $\mathbf{C}^n$. The 5-dimensional examples $H00$–$H14$ were given in [12] as examples that are not Strongly Nilpotent. Example $H00$ has the distinction that its bilinear $B(x,y)$-matrix is not identically nilpotent.

TABLE 3. Degrees of Non-cubic-linear Examples

| *Dim & Name* | $\deg f$ | $\deg f^{-1}$ | $\deg h_s$ | $\deg h_s^{-1}$ | Comments |
|---|---|---|---|---|---|
| 2D-Rusek [21] | 4 | 4 | infinite | infinite | $x + nonhomog$. See §§ 6 & 6.1 |
| 5D-ROD-Rusek | 3 | 7 | 7 | 7 | Reduced to cubic-hom. See § 6.2 |
| 3D power-exact [16] | 3 | 5 | 5 | 5 | $g'(x)$ & $g'(x)^2$ both exact |
| 5D-H00(3.4) | 3 | 5 | 5 | 5 | Not S.N. & $B(x,y)^n \not\equiv 0$ |
| 5D-H01(3.4) | 3 | 5 | 5 | 5 | Not S.N. & $B(x,y)^4 \equiv 0$ |
| 5D-H02(3.4) | 3 | 7 | 7 | 7 | ditto |
| 5D-H03(3.4) | 3 | 7 | 7 | 7 | ditto |
| 5D-H04(3.4) | 3 | 7 | 7 | 7 | See Sec. 7. Not S.N. |
| 5D-H05(3.4) | 3 | 7 | 7 | 7 | Not S.N. & $B(x,y)^4 \equiv 0$ |
| 5D-H06(3.4) | 3 | 7 | 7 | 7 | ditto |
| 5D-H07(3.4) | 3 | 7 | 7 | 7 | ditto |
| 5D-H08(4.5) | 3 | 13 | 13? | Incomplete | Not S.N. & $B(x,y)^5 \equiv 0$ |
| 5D-H09(4.5) | 3 | 13 | 13? | ditto | ditto |
| 5D-H10(4.5) | 3 | 15 | 15? | ditto | ditto |
| 5D-H11(4.5) | 3 | 19 | 19? | ditto | ditto |
| 5D-H12(4.5) | 3 | 19 | 19? | ditto | ditto |
| 5D-H14(4.5) | 3 | 21 | 21? | ditto | ditto |

## 7. Another cubic-homogeneous but not cubic-linear

$y = f(x) = x - g(x) \neq x - [diag(Ax)]^2 Ax$ , $g(tx) = t^3 g(x)$, and $g'(x)^n = 0$

**5D-H04(3.4)** This example is cubic-homogeneous, but not cubic-linear, not strongly nilpotent, and not triangularizable. It, and all $H$-examples in Table 3 , are from [12].

Degrees $\{f,\ f^{-1},\ h_s,\ h_s^{-1}\} = \{3,\ 7,\ 7,\ 7\}$, $g'(x)^4 = 0$, $g(g(x)) = 0$, $B(x, g(x))^2 = 0$, $B(x,y)^4 = 0$, and $g'(x)$ is not S.N.

$$f(x) = \begin{bmatrix} x_1 \\ x_2 - {x_1}^2\, x_3 - c\, {x_5}^3 \\ x_3 - {x_1}^2\, x_4 - x_2\, {x_5}^2 \\ x_4 - {x_1}^2\, x_5 + x_3\, {x_5}^2 \\ x_5 \end{bmatrix}$$

$$f^{-1}(y) = \begin{bmatrix} y_1 \\ y_2 + {y_1}^2\, y_3 + {y_1}^4\, y_4 + {y_1}^6\, y_5 + {y_1}^2\, y_2\, {y_5}^2 + c\, {y_5}^3 + c\, {y_1}^2\, {y_5}^5 \\ y_3 + {y_1}^2\, y_4 + {y_1}^4\, y_5 + y_2\, {y_5}^2 + c\, {y_5}^5 \\ y_4 + {y_1}^2\, y_5 - y_3\, {y_5}^2 - {y_1}^2\, y_4\, {y_5}^2 - {y_1}^4\, {y_5}^3 - y_2\, {y_5}^4 - c\, {y_5}^7 \\ y_5 \end{bmatrix}$$

$h_s(x) =$

$$\begin{bmatrix} x_1 \\ x_2 + \frac{s^2\, {x_1}^2 \left({x_1}^2\, x_4 + x_2\, {x_5}^2\right)}{(s-1)^2 (s+1)^2 (s^2+1)} + \frac{{x_1}^2\, x_3 + c\, {x_5}^3}{(s^2-1)} + \frac{s^6 \left({x_1}^6\, x_5 + c\, {x_1}^2\, {x_5}^5\right)}{(s-1)^3 (s+1)^3 (s^2+1)(s^2-s+1)(s^2+s+1)} \\ x_3 + \frac{{x_1}^2\, x_4 + x_2\, {x_5}^2}{s^2-1} + \frac{s^2 \left({x_1}^4\, x_5 + c\, {x_5}^5\right)}{(s-1)^2 (s+1)^2 (s^2+1)} \\ x_4 + \frac{x_5 \left({x_1}^2 - x_3\, x_5\right)}{s^2-1} - \frac{s^2\, {x_5}^2 \left({x_1}^2\, x_4 + x_2\, {x_5}^2\right)}{(s-1)^2 (s+1)^2 (s^2+1)} + \frac{s^6 \left(-\left({x_1}^4\, {x_5}^3\right) - c\, {x_5}^7\right)}{(s-1)^3 (s+1)^3 (s^2+1)(s^2-s+1)(s^2+s+1)} \\ x_5 \end{bmatrix}$$

$h_s^{-1}(y) =$

$$\begin{bmatrix} y_1 \\ \left\{ \begin{array}{l} y_2 + \frac{{y_1}^4 y_4}{(s-1)^2 (s+1)^2 (s^2+1)} - \frac{{y_1}^6 y_5}{(s-1)^3 (s+1)^3 (s^2+1)(s^2-s+1)(s^2+s+1)} - \frac{c{y_5}^3}{s^2-1} \\ -\frac{{y_1}^2 y_3}{s^2-1} + \frac{{y_1}^2 y_2 {y_5}^2}{(s-1)^2 (s+1)^2 (s^2+1)} - \frac{c{y_1}^2 {y_5}^5}{(s-1)^3 (s+1)^3 (s^2+1)(s^2-s+1)(s^2+s+1)} \end{array} \right\} \\ y_3 - \frac{{y_1}^2 y_4}{s^2-1} + \frac{{y_1}^4 y_5}{(s-1)^2 (s+1)^2 (s^2+1)} - \frac{y_2 {y_5}^2}{s^2-1} + \frac{c{y_5}^5}{(s-1)^2 (s+1)^2 (s^2+1)} \\ \left\{ \begin{array}{l} y_4 + \frac{y_3 {y_5}^2}{s^2-1} + \frac{{y_1}^4 {y_5}^3}{(s-1)^3 (s+1)^3 (s^2+1)(s^2-s+1)(s^2+s+1)} - \frac{y_2 {y_5}^4}{(s-1)^2 (s+1)^2 (s^2+1)} \\ + \frac{c{y_5}^7}{(s-1)^3 (s+1)^3 (s^2+1)(s^2-s+1)(s^2+s+1)} + \frac{{y_1}^2 y_5 (1-s^4-y_4 y_5)}{(s-1)^2 (s+1)^2 (s^2+1)} \end{array} \right\} \\ y_5 \end{bmatrix}$$

## 8. Conjectures: What seems to be true

For each example we computed (except for Rusek's discussed in § 6), the mapping $h_s$ defined by the (inverse) Schröder Eq. (1) turned out to be a *polyomorphism* of $\mathbf{C}^n$ and a *conjugation* of $sf(x)$ to $sx$; not merely a *holomorphic* mapping or a *formal power series*. (The equation $F \circ H = H \circ L$ is sometimes called **Schröder's Equation**; and $H$ is called **Schröder's map**. Our Eq. (1) has the form $h \circ F = L \circ h$, so our $h$ is $H^{-1}$.) Indeed, each polyomorphism $f : \mathbf{C}^n \to \mathbf{C}^n$, with $f(0) = 0$ and $f'(0) = I$, seems to have an associated homotopy $h : \mathbf{C}_s \times \mathbf{C}_x^n \to \mathbf{C}^n$ such that, for all but finitely many complex numbers $s$, $h_s$ is an automorphism of $\mathbf{C}^n$ and conjugates $sf(x)$ to $sx$ by means of the Eq. (1). Each $h_s$ is at least a formal power series in the variables $x = (x_1, \ldots, x_n)$, uniquely determined by Eq. (1) with $h_s'(0) = I$. Furthermore, in many cases (e.g., for all polynomial maps of the form $f(x) = x +$ *cubic-homogeneous*) $h_s(x)$ is actually a polyomorphism of $\mathbf{C}^n$ which is defined *for all but finitely many complex numbers $s$ on the* **unit circle**, and we always interpret the limits

$$f^{-1}(x) = \lim_{s \to 0} s\, h(1/s, t/s) = \lim_{t \to \infty} \frac{1}{t} h(t, tx)$$

in *3*(b) of Prop. 1 to mean the *term-by-term* limit of the series. Thus, — If $g(x)$ is a cubic-homogeneous polynomial mapping of $\mathbf{C}^n$ into itself with nilpotent Jacobian, then we add to the following four older conjectures

1. $g(\mathbf{C}^n)$ is contained in a proper linear subspace of $\mathbf{C}^n$,
2. Some iterate of $g$ is identically zero,
3. The Jacobian (derivative) matrix of every iterate of $g$ is nilpotent,
4. If $\mathcal{B}(x, y) := g'\left(\frac{x+y}{2}\right) - g'\left(\frac{x-y}{2}\right)$, then $\mathcal{B}(u, v)(u - v) = (u - v)$ only for $u = v$,

the following **linearization conjecture**:

5. To the mapping $f(x) = x - g(x)$ there corresponds a one-parameter family of *polynomial* maps $x \mapsto h_s(x)$ which satisfies the (inverse) Schröder Eq. (1) for all $x$ in $\mathbf{C}^n$ and *for all but finitely many complex numbers $s$ on the unit circle*. Furthermore, each (defined) $h_s$ is itself a polyomorphism of $\mathbf{C}^n$ which conjugates $sf(x)$ to $sx$; and satisfies $h_0(x) = f(x)$ and $h_\infty(x) = x$ for all $x$ in $\mathbf{C}^n$. In addition, we conjecture that $deg\, h_s^{-1} = deg\, h_s = max\{deg\, f, deg\, f^{-1}\}$.

## 9. The Poincaré-Siegel Theory

The **Poincaré-Siegel Theory of Linearization and Non-Resonance** ([1, Chap. 5] & [18, § 3]) implies dilated polyomorphisms $sf(x) = sx - sg(x)$ are *analytically* conjugate to their linear part at least *locally*. That they are *globally* and *polyomorphically* so, at least when $g(x)$ is *homogeneous* with *nilpotent* Jacobian, is indicated by the results presented in this paper. We briefly summarize the Poincaré-Siegel Theory in order to show its connection with conjugations of dilated polyomorphisms. See also [7, 20, 23, 24].

*Definition of Resonances.* The $n$-tuple $s = (s_1, \ldots, s_n)$ of **eigenvalues** of a linear map $L$, such as the derivative map $x \mapsto f'(0)x$ at the fixed-point $x = 0$ of a mapping $f(x)$, is said to be **resonant** if one of these eigenvalues (say $s_k$) satisfies: $s_k = s^m = s_1^{m_1} \cdots s_n^{m_n}$, for integers $m_j \geq 0$ with $|m| = m_1 + \cdots + m_n \geq 2$. This is called a **resonance** of **order** $|m|$.

*Poincaré's Theorem.* Let $f$ be analytic at $x = 0$, $f(0) = 0$, and assume the eigenvalues $s = (s_1, \ldots, s_n)$ of $f'(0)$ are not in resonance (of any order $|m|$); then there exists a formal power series $h(x)$ so that

$$h^{-1}(f(h(x))) = f'(0)x.$$

*Siegel's Theorem.* Let $f$ be analytic at $x = 0$ with $f(0) = 0$. If the eigenvalues $s = (s_1, \ldots, s_n)$ of $f'(0)$ are of **multiplicity-type** $(C, \nu)$, for some positive constants $C$ and $\nu$, in the sense that

$$|s_j - s^m| \geq C/|m|^\nu$$

for all $m = (m_1, \ldots, m_n)$ with $|m| \geq 2$ and $j \in \{1, 2, \ldots, n\}$, then the formal power series for $h$ and $h^{-1}$ which occur in Poincaré's Theorem *converge* near $x = 0$.

*The Special Case of Dilated Polyomorphisms.* For each complex $s$, the eigenvalues of the linear map $x \mapsto sf'(0)$ are all equal to $s$. Then Siegel's inequality $|s_j - s^m| \geq C/|m|^\nu$ is satisfied for some constants $C$ and $\nu$.

## Appendix

### A. Nelson's example, Anick's example & Reduction of Degree

A.1. NELSON'S EXAMPLE OF THE POINCARÉ-SIEGEL THEORY

Nelson [18, page 32] gives the following 2-dimensional example to illustrate the fact that it is not always possible to choose coordinates at a fixed point of a vector field so that it becomes locally linear there. But his example is a family of *polyomorphisms* which is *generically* even *globally* conjugate to its linear part;—in both ways of considering it:

A.1.1. *First, as Discrete Dynamical Systems (iteration).*
Nelson's example is the mapping

$$\begin{bmatrix} u \\ v \end{bmatrix} = f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a\,x + c\,y^2 \\ b\,y \end{bmatrix},$$

with $abc \neq 0$.

Its inverse is easily found to be

$$\begin{bmatrix} x \\ y \end{bmatrix} = f^{-1}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} (u/a) - (c\,v^2)/(a\,b^2) \\ v/b \end{bmatrix}.$$

We can find $h = (p, q)$ satisfying $h \circ f = L \circ h$ by repeatedly differentiating

$$h\left(\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}\right) = \begin{bmatrix} p(u,v) \\ q(u,v) \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} p(x,y) \\ q(x,y) \end{bmatrix} = \begin{bmatrix} a\,p(x,y) \\ b\,q(x,y) \end{bmatrix} \qquad (6)$$

and evaluating at $(x, y) = (0, 0)$ with $h(0) = 0$ and $h'(0) = I$. Thus we find

$$h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + c\,y^2/(a - b^2) \\ y \end{bmatrix} \quad , \quad h^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - c\,y^2/(a - b^2) \\ y \end{bmatrix}.$$

If $a = b^2$, $h$ does not exist even locally. Otherwise (6) holds *globally*!

A.1.2. *Second, as continuous dynamical systems (vector fields).*
The vector field $(\dot{x}, \dot{y}) = (u, v) = (a\,x + c\,y^2, b\,y)$ is linearized to

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} p_x & p_y \\ q_x & q_y \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} p_x & p_y \\ q_x & q_y \end{bmatrix}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} a\,p \\ b\,q \end{bmatrix} \qquad (7)$$

by differentiating (7) repeatedly and evaluating at $(0, 0)$ with $R(0) = 0$ & $R'(0) = I$, where $R = (p, q)$. Thus we obtain

$$R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + c\,y^2/(a - 2b) \\ y \end{bmatrix}.$$

One easily checks that, with this $R$, (7) holds *globally* for $a \neq 2b$; but not even locally when $a = 2b$, which is Nelson's point. Moreover, both of the coordinate-changes $h$ and $R$ are *polyomorphisms* of $\mathbf{C}^2$, except at the resonances ($a = b^2$ for $h$ and $a = 2b$ for $R$);—not merely *analytic* automorphisms.

A.2. ANICK'S 4D CUBIC-HOMOGENEOUS NON-TRIANGULARIZABLE

**4D-A(2.3)** This example of a non-triangularizable, not-known-to-be-tame but stably tame, cubic-homogeneous map occurs in David Wright's paper [25]; and is also discussed in Hubbers' Thesis [8, § 1.5]. But its source is evidently the paper by Martha K. Smith [Stably Tame Automorphisms, *Journal of Pure and Applied Algebra* **58** (1989) 209–212] where it is called a (previously) unpublished example of David Anick. This example satisfies $y = f(x) = x - g(x) \neq x - [diag(Ax)]^2 Ax$ , $g(tx) = t^3 g(x)$, and $g'(x)^n = 0$.

Degrees $\{f,\ f^{-1},\ h_s,\ h_s^{-1}\} = \{3,\ 3,\ 3,\ 3\}$, $g'(x)^3 = 0$, $g(g(x)) = 0$, $B(x, g(x))^2 = 0$, $B(x, y)^4 \neq 0$, and $g'(x)$ is not S.N.

$$f(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 - x_1\, x_2\, x_3 - {x_2}^2\, x_4 \\ x_4 + x_1\, x_2\, x_4 + {x_1}^2\, x_3 \end{bmatrix} \qquad f^{-1}(y) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 + y_1\, y_2\, y_3 + {y_2}^2\, y_4 \\ y_4 - y_1\, y_2\, y_4 - {y_1}^2\, y_3 \end{bmatrix}$$

$$h_s(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + x_2\ (x_1\, x_3 + x_2\, x_4)\,/(s^2 - 1) \\ x_4 - x_1\ (x_1\, x_3 + x_2\, x_4)\,/(s^2 - 1) \end{bmatrix}$$

$$h_s^{-1}(y) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 - y_2\ (y_1\, y_3 + y_2\, y_4)\,/(s^2 - 1) \\ y_4 + y_1\ (y_1\, y_3 + y_2\, y_4)\,/(s^2 - 1) \end{bmatrix}$$

It is interesting to note that although the matrix $g'(x) \equiv B(x, x) =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_2\, x_3 & x_1\, x_3 + 2\, x_2\, x_4 & x_1\, x_2 & {x_2}^2 \\ -2\, x_1\, x_3 - x_2\, x_4 & -x_1\, x_4 & -{x_1}^2 & -x_1\, x_2 \end{bmatrix}$$

is nilpotent for all $x$, the bilinear matrix $B(x, y) =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{x_3\, y_2 + x_2\, y_3}{2} & \frac{x_3\, y_1 + 2\, x_4\, y_2 + x_1\, y_3 + 2\, x_2\, y_4}{2} & \frac{x_2\, y_1 + x_1\, y_2}{2} & x_2\, y_2 \\ \frac{-2\, x_3\, y_1 - x_4\, y_2 - 2\, x_1\, y_3 - x_2\, y_4}{2} & \frac{-x_4\, y_1 - x_1\, y_4}{2} & -x_1\, y_1 & \frac{-x_2\, y_1 - x_1\, y_2}{2} \end{bmatrix}$$

is *not* nilpotent for some $x \neq y$. For it was proved in [12, § 3.3] that if we define $\mathcal{B}(x, y) := g'\left(\frac{x+y}{2}\right) - g'\left(\frac{x-y}{2}\right)$, then the mapping $f(x) = x - g(x)$ is injective if and only if $\mathcal{B}(u, v)(u - v) = (u - v)$ only for $u = v$. As Anick's example illustrates, *nilpotence* of $\mathcal{B}(x, y)$ is not *necessary*, although it is clearly *sufficient*, for the latter condition to hold.

A.3. REDUCTION ALGORITHM

Here is the algorithm used to "reduce" the (dimension-2, degree-4) map in § 6.2 to a (dimension-5, degree-3) map of the special form $x$ + *cubic-homogeneous*. It is based on what I learned from Arno van den Essen and David Wright. It is an algorithm to transform a polynomial Keller map $f : C^n \to C^n$ to one of the form $F : C^{(n+m)} \to C^{(n+m)}$ with $m \geq 1$, $F(x) = x - g(x)$, $g(tx) = t^3 g(x)$, and $\det F'(x) = 1$. Given a Keller map $f(x)$, we first get it into the form $x + \cdots$ by subtracting $f(0)$ and multiplying by $[f'(0)]^{-1}$. Then apply the following processes to *each component* of the mapping $f(x)$ as often as necessary:

**Process I**. **Reduction to the form** $x + [\textit{terms of degrees } 2 \textit{ and } 3 \textit{ only}]$.
**Case 1**. If a component $f_i$ (assumed here to be $f_1$) contains a perfect square $S(x)^2$ of degree $> 3$, then write $f(x, u) = (f(x), u)$ and $F(x, u) :=$

$$G(f(H(x,u))) = \Big(f_1(x) - (u + S(x))^2, f_2(x), \ldots, f_n(x), u + S(x)\Big),$$

where $G = (x_1 - u^2, x_2, \ldots, x_n, u)$ and $H = (x_1, \ldots, x_n, u + S(x))$.
**Case 2**. If a component $f_i$ (assumed here to be $f_1$) of degree $> 3$ does not contain a perfect square, then let $M$ be a homogeneous term in it of degree $> 3$ that factors as $M = PQ$ with $degP = 2$. (This is always possible by expanding $f_i$ if necessary into a sum of monomials.) In this case write $f(x, u, v) = (f(x), u, v)$ and $F(x, u, v) := G(f(H(x, u, v))) =$

$$(f_1(x) - (u + P(x))(v + Q(x)), f_2(x), \ldots, f_n(x), u + P(x), v + Q(x)),$$

where $G = (x_1 - uv, x_2, \ldots, x_n, u, v)$ and $H = (x_1, \ldots, x_n, u + P, v + Q)$. Note that $S$ (or $M$) has been eliminated from $F$ and that both $G$ and $H$ are polyomorphisms with unit Jacobian determinant; therefore $\det F' = \det[(G \circ f \circ H)'] = \det f'(H)$. Furthermore, $F$ is a polyomorphism iff $f$ is.

**Process II**. **Homogenization to the form** $x + [\textit{terms of degree } 3 \textit{ only}]$. Replace each cubic term in each component of $F$ by (*minus new variable*) and tack on at the end an additional component: [(*this new variable*) + (*that cubic term*)]. Then tack on a final new variable $t$ as an additional component at the end, and multiply each term in all other components (other than the leading linear term $x_i$) by a power of $t$ necessary to make it cubic. This introduces $c + 1$ new dimensions where $c$ is the number of components of $F$ that contain cubics.
**EXAMPLE #1**: The map $f(x, y) = (x + (y + x^2)^2, y + x^2)$ reduces to

$$F(x, y, u, v, t) = (x + t(y^2 - u^2 - tv), y + tx^2, u + tx^2, v + 2x^2(y - u), t);$$

or to the *different* cubic-homogeneous mapping (reduction is not unique)

$$F(x, y, u, v, t) = (x - t(2uy + u^2 + tv), y + tx^2, u + tx^2 + ty^2, v - 2ux^2, t).$$

**EXAMPLE #2: Nagata's conjectured non-tame polyomorphism**

$$f = (x - 2y(xz + y^2) - z(xz + y^2)^2, y + z(xz + y^2), z).$$

Using the above Reduction Process I (case #2) applied twice: First with $P = -(xz + y^2)$ and $Q = (xz + y^2)z$, and second with $P = -(xz + y^2)$ and $Q = zu$; followed by the Homogenization Process II with new variables $w$, $p$, $q$, $t$; we obtain the cubic-homogeneous polyomorphism of $\mathbf{C}^{11}$

$$F = (x - wt^2 - uvt - rst, y - pt^2, z, u - t(xz + y^2), v - qt^2, r - t(xz + y^2),$$

$$s + zut, w + (s + v - 2y)(xz + y^2) - zur, p + z(xz + y^2), q + z(xz + y^2), t).$$

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The author thanks Bo Deng and Gaetano Zampieri for pointing out to him (at lunch in the Colonial Room of the University of Nebraska Union on Thursday, March 3, 1994) that *dilated* polyomorphisms $x \mapsto sf(x)$, where $f(0) = 0$ and $f'(0) = I$, might be *globally* conjugate to the dilation $L_s(x) = sx$. That is, since the Poincaré-Siegel Theory for *analytic* maps $f = x + \cdots$ says there exists *locally* (in a neighborhood of the fixed-point $x = 0$) an *analytic* map $H : \mathbf{C}^n \to \mathbf{C}^n$ satisfying the equation $sf = H \circ L_s \circ H^{-1}$; such a conjugation map might exist *globally* for *polyomorphisms* $f(x) = x - g(x)$. Indeed, such an $H$ does exist at least *formally*. When is it an entire function? When is it a polynomial? When is it a polyomorphism? At that time they had not computed any examples, and I was doubtful that such conjugations could be *global*. But the examples of $h_s$ presented in this paper (which I computed in response to their comments) show that an $h_s$ satisfying Eq. (1) does exist globally as a *polyomorphism* in many cases when $f(x)$ has the special form $f(x) = x - g(x)$ with $g(tx) = t^3 g(x)$ and $g'(x)^n = 0$. At van den Essen's Curaçao Conference, July 4–8, 1994, I offered \$100 (U.S.) to the first person to show me a counterexample of this type. *On Monday, September 19, 1994, I found two e-mail messages from Arno van den Essen, dated September 16 and 19, in which he described two counterexamples!* On September 19 he FAXed me a proof that the $h(s, x)$ defined by Eq. (1) for $f(x) = (x_1 + p(x)x_4, x_2 - p(x)x_3, x_3 + {x_4}^3, x_4)$, where $p(x) = x_3x_1 + x_4x_2$, can not be polynomial! It still remains to answer the questions: Is the $h(s, x)$ for this example *entire*? Will each $h_s(x)$ be a polyomorphism (or at least a holomorphic automorphism of $\mathbf{C}^n$) if $f(x)$ is *cubic-linear*?

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## References


1. V. I. Arnold. Geometrical Methods in the Theory of Ordinary Differential Equations, vol. 250 *Grundlehren Math. Wiss.* Springer-Verlag, New York Heidelberg Berlin 1983. Chapter 5: Resonances, Poincaré's Theorem, and Siegel's Theorem.
2. H. Bass, E. H. Connell, and D. Wright. The Jacobian Conjecture: Reduction of Degree and Formal Expansion of the Inverse. *Bull. Amer. Math. Soc.* **7**(2) (1982).
3. H. Bass and G. H. Meisters. Polynomial Flows in the Plane. *Adv. in Math.* **55** (1985) 173–208.
4. A. Białynicki-Birula and M. Rosenlicht. Injective morphisms of real algebraic varieties. *Proc. Amer. Math. Soc.* **13** (1962) 200–203.
5. B. Deng, G. H. Meisters and G. Zampieri. Conjugation for polynomial mappings. Preliminary Manuscript May 1994.
6. L. M. Drużkowski. An Effective Approach to Keller's Jacobian Conj. *Math. Ann.* **264** (1983) 303–313.
7. S. Friedland and J. Milnor. Dynamical properties of plane polynomial automorphisms. *Ergodic Theory and Dynamical Systems* **9** (1989) 67–99.
8. E. Hubbers. The Jacobian Conjecture: Cubic Homogeneous Maps in Dimension 4. Masters Thesis directed by Arno van den Essen, Catholic University of Nijmegen, The Netherlands, February 17, 1994.
9. H. W. E. Jung. Über ganze birationale Transformationen der Ebene. *J. Reine Angew. Math.* **184** (1942) 161–174.
10. Ott-Heinrich Keller [22.VI.1906–5.XII.1990]. Ganze Cremona Transformationen. *Monatsh. Math.* **47** (1939) 299–306. See items 6 & 7 in Keller's table on page 301.
11. W. van der Kulk. On polynomial rings in two variables. *Nieuw Arch. Wisk.* **(3) I** (1953) 33–41.
12. G. H. Meisters. Inverting polynomial maps of $n$-space by solving differential equations. In *Delay and Differential Equations, Proceedings in Honor of George Seifert, Ames, Iowa, Oct. 18–19, 1991*, eds. Fink-Miller-Kliemann, pp. 107–166, World Sci. Pub. Singapore • Teaneck NJ • London • Hong Kong, 1992. ISBN 981–02–0891-X.
13. G. H. Meisters. Power Similarity: Summary of First Results. **Conference on Polynomial Automorphisms** at C. I. R. M. Luminy, France, Oct. 12–17, 1992.
14. G. H. Meisters. Invariants of Cubic-Similarity. In *Recent Results on the Global Asymptotic Stability Jacobian Conjecture*, ed. Marco Sabatini, Trento, Italy, Sept. 14–17, 1993. Dipartimento di Matematica, Università di Trento, I–38050.
15. G. H. Meisters and Cz. Olech. Strong Nilpotence Holds in Dimensions up to Five Only. *Linear and Multilinear Algebra* **30** (1991) 231–255. **MR 92i**:15009.
16. G. H. Meisters and Cz. Olech. Power-Exact, Nilpotent, Homogeneous Matrices. *Linear and Multilinear Algebra* **35** (Nos. 3–4) (1993) 225–236.
17. M. Nagata. *On Automorphism Group of* **k**[x,y], vol. **5** of *Lectures in Mathematics, Kyoto University.* Kinokuniya Bookstore, Tokyo, Japan, 1972.
18. E. Nelson. Topics in Dynamics I: Flows. Princeton University Press 1970. Section 3.
19. D. J. Newman. One-one polynomial maps. *Proc. A. M. S.* **11** (1960) 867–870.
20. J.-P. Rosay and W. Rudin. Holomorphic maps from $\mathbf{C}^n$ to $\mathbf{C}^n$. *Trans. A.M.S.* **310** (1988) 47–86.
21. K. Rusek. A Geometric Approach to Keller's Jacobian Conjecture. *Math. Ann.* **264** (1983) 315 - 320.
22. K. Rusek. Polynomial Automorphisms. IMPAN Preprint **456** 1989. Math Inst, Polish Academy of Sciences, ul. Reymonta 4, PL-30–059 Kraków, Poland.
23. C. L. Siegel. Iteration of Analytic Functions. *Ann. of Math.* **43**(4) (1942) 607–612.
24. S. Sternberg. Local contractions, a theorem of Poincaré, and the structure of local homeomorphisms: I, II, III. *Amer. J. Math.* **79** (1957) 809–824; **80** (1958) 623–631; **81** (1959) 578–604.
25. D. Wright. The Jacobian Conjecture: Linear Triangularization for Cubics in Dimension Three. *Linear and Multilinear Algebra* **34** (1993) 85–97.