The Cubic-Linear Linearization Conjecture

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1 Keller's Jacobian Conjecture Rephrased

In § 2 below I state a question, still open, I raised at Curaçao [7] in July 1994. But first some terminology and background. For vectors x in $\mathbb{C}^n$, let $diag(x)$ denote the diagonal matrix whose diagonal entries are the components of the vector x. A given $n \times n$ complex matrix A serves as the **kernel matrix** for the matrix-valued bilinear function of the two vector variables $x, y \in \mathbb{C}^n$:

$$\mathcal{B}_A(x, y) := 3[diag(Ax)][diag(Ay)]A. \tag{1}$$

Open question: Is there an $n \times n$ matrix A satisfying both of the following conditions?

Cond 1. The matrix $\mathcal{B}_A(x, x)$ is nilpotent for all x in $\mathbb{C}^n$. (A is **Keller admissible**.)

Cond 2. There are distinct vectors x and y in $\mathbb{C}^n$ such that $\mathcal{B}_A(x, y)(x - y) = (x - y)$.

In [6] I called a complex matrix A satisfying both of these conditions a **bad** matrix.

Note 1. The matrix-valued bilinear function $\mathcal{B}_A(x, y)$ has the following properties:

(a) $\mathcal{B}_A(x, y) = \mathcal{B}_A(y, x)$ for all vectors x, y;

(b) $\mathcal{B}_A(x, y)z = \mathcal{B}_A(x, z)y$ for all vectors x, y, z;

(c) $\mathcal{B}_A(x, x)$ is the Jacobian matrix $H_A'(x)$ of the cubic-homogeneous mapping $H_A(x) := [diag(Ax)]^3\mathbf{1}$, where $\mathbf{1}$ denotes the column $[1, 1, \dots, 1]^T$.

Note 2. Ott-Heinrich Keller [4] essentially asked the question: If $\det[F'(x)] \equiv 1$ for a polynomial mapping F, is then F bijective with polynomial inverse? It suffices [1, 8] to prove injectivity. It even suffices [2] to prove injectivity for the special **cubic-linear** maps $F_A(x) := x - H_A(x) := x - [diag(Ax)]^2Ax$. It was proved in [5, §3.3 p. 118] that a poly map $F(x) := x - H(x)$, with $H(tx) = t^3H(x)$, is injective if and only if Cond 2 is *false*. This is a direct consequence of the identity

$$F(x) - F(y) = (i\sqrt{3})[I - \mathcal{B}(u, v)](u - v)$$

which is the **mean-value formula**

$$F(x) - F(y) = [I - (1/3)\{\mathcal{B}(x, x) + \mathcal{B}(x, y) + \mathcal{B}(y, y)\}](x - y)$$

with the transformation $x = au + \bar{a}v$ and $y = \bar{a}u + av$, where $a = (1 + \sqrt{3}i)/2$. Cond 2 is clearly false if $\mathcal{B}_A(x, y)$ is *nilpotent for all* x, y; for then all eigenvalues of $\mathcal{B}_A(x, y)$ must be *zero* for all x, y. Cond 1 (*admissibility*) is equivalent to $\det[F_A'(x)] \equiv 1$, which is necessary for F_A to be injective. See [5, Lemma 1(c) page 112 & Eq (2.2) page 110]. Thus the above open question is just Keller's question rephrased: An answer "yes" to our question is "no" to Keller's.

2 Linearization Conjecture for Cubic-Linear Maps

The examples given in [7] suggest that perhaps **The Cubic-Linear Linearization Conjecture** is true: Namely, to each cubic-linear mapping

$$F_A(x) := x - H_A(x) := x - [diag(Ax)]^2 Ax = x - (1/3)\mathcal{B}_A(x,x)x, \tag{2}$$

satisfying Condition 1, there corresponds a (normalized) 1-parameter family h_s of *homeomorphisms* $x \mapsto h_s(x) \equiv h(s,x)$ of $\mathbb{C}^n$ which satisfies the Schröder Equation

$$h(s, s\,F_A(x)) = s\,h(s,x), \text{ with } h(s,0) = 0 \text{ and } \partial_x h(s,0) = I, \tag{3}$$

for all x in $\mathbb{C}^n$ and for all complex numbers s not on the unit circle; and which therefore conjugates $sF_A(x)$ to sx. In fact, each h_s found in [7] is actually a *polyomorphism*[1] of $\mathbb{C}^n$ for all but finitely many complex numbers s on the unit circle. Perhaps each Schröder function $h_s(x)$, defined by (3) as a formal power series in x for each $|s| \neq 1$, is at least a *holomorphic automorphism* of $\mathbb{C}^n$. Furthermore, it seems that one always has $h_0(x) = F_A(x)$ and $h_\infty(x) = x$ for all x in $\mathbb{C}^n$.

If this conjecture is true, then Keller's Jacobian Conjecture is true. About two months after the July 1994 Curaçao Conference, Arno van den Essen produced a counter-example [3] to the *stronger* conjecture (for which I had offered \$100 at Curaçao) that the above statement might be true for the even wider class of **cubic-homogeneous** polynomial maps $F(x) = x - H(x)$ where $H(tx) = t^3 H(x)$. Arno collected the \$100 by showing that $h(s,x)$, defined by (3) for $F(x) = (x_1 + p(x)x_4, x_2 - p(x)x_3, x_3 + {x_4}^3, x_4)$ where $p(x) = x_3x_1 + x_4x_2$, *can not be polynomial*! Even if this **Scalar-s Cubic-Linear Linearization Conjecture** is not true, there is still the more general (weaker) possibility that the **Matrix-S Cubic-Linear Linearization Conjecture** is true, where the scalar s is replaced by an invertible matrix S with appropriate eigenvalues. At least we would like to know: *For which matrices A does there exist such a matrix S?*

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[1] *polyomorphism* is short for *polynomial automorphism*.