Matrix Model Exercises — due June 22 and 23

Work the problems on scratch paper. Be prepared to ask questions in class on June 22 regarding any problems that you aren't getting.

You may discuss the problems with each other verbally. Please do not share any written work.

Don't turn in your scratch paper. Instead, write out the solutions neatly (by hand is fine) on paper that doesn't have torn edges. If your solutions require more than one page, attach the pages together with a staple or paper clip. The written solutions are due on June 23.

Your solutions should include some writing, as necessary. This is not as formal as writing a paper, but you still should have enough writing to explain anything that is not obvious. For example, your solution to problem 2 should begin with the general equation,

$$\det(\mathbf{M} - \lambda \mathbf{I}) = 0,$$

which you solve to determine the eigenvalues; your solution to problem 3 requires a small explanation of the logic used to solve the problem.

Note that the polynomial defined by $\det(\mathbf{M} - \lambda \mathbf{I})$ is called the **characteristic polynomial** for the matrix $\mathbf{M}$.

1. Some authors use t instead of n for the discrete time variable. It is also common for the time to be written as a subscript rather than as a function argument in parentheses. Write the model

$$p_{t+1} = ap_t + bq_t + cr_t, \qquad q_{t+1} = dp_t, \qquad r_{t+1} = fq_t$$

in matrix-vector form. Note that any quantities without subscripts are assumed to be positive parameters.

2. Find the eigenvalues for each of the following matrices:

(a) $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ (b) $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$

3. Suppose $\lambda = 1$ is an eigenvalue of the matrix $\mathbf{L} = \begin{pmatrix} f_1 & f_2 \\ p_1 & 0 \end{pmatrix}$. Determine a relation that the parameters must satisfy.

4. Given $s = \frac{1}{12}$, $a = \frac{1}{2}$, and $p = \frac{2}{3}$ in our boxbug model, determine how large f must be so that $\lambda \geq 1$.

5. The system $\mathbf{x}(n+1) = \mathbf{L}\mathbf{x}(n)$, where $\mathbf{L} = \begin{pmatrix} 0 & 9 & 12 \\ 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$, represents a stage-structured population model for some animal species.

   (a) Sketch a life-cycle graph showing the possible transitions from one stage to another.

   (b) Suggest a life history that corresponds to the model. (Hint: It is *not* larvae-pupa-adult.)

   (c) Determine the characteristic polynomial.

   (d) Find the long-term growth rate by graphing the characteristic polynomial for $\lambda > 0$.

6. Find an eigenvector for the largest positive eigenvalue for each of the matrices in Exercise 2.

7. Find the long-term population age distribution (as fraction of the total) for the system of Exercise 5.

8. Consider a matrix population model with

$$\mathbf{M} = \begin{pmatrix} 0 & f_2 & f_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{pmatrix}.$$

   (a) Find the characteristic equation.

   (b) Given $f_2 = 2$, $f_3 = 5$, and $s_2 = 0.8$, determine how large s_1 must be to get $\lambda \geq 1$.

   (c) Given $f_2 = 2$, $f_3 = 5$, $s_1 = 0.2$, and $s_2 = 0.8$, find λ and the long-term population ratios with three-decimal-digit accuracy. (Hint: Modify the Boxbug program we wrote on 6/19.)