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Math 221, section 2 **Preliminary Exam** 4/28/99

Name [redacted]

(10) 1. Find the form of the particular solution for the problem

$$y'' - y = te^t + e^t \cos t.$$

DO NOT EVALUATE THE COEFFICIENTS.

Check y_c.

$$\boxed{y_p = \cancel{Ate^t} + (A\cos t + B\sin t)e^t}$$

4

$te^t \Rightarrow y_p = Ate^t$ ①

$e^t\cos t \Rightarrow y_p = (A\cos t + B\sin t)e^t$ ②

$y_p =$ ① + ②

(16) 2. Find the general solution of the problem

$$y'' + 2y' + 5y = e^{-2t}.$$

$\lambda = -\frac{2}{2} \pm \frac{\sqrt{4-20}}{2}$

$\lambda = -1 \pm 2i$

$y_c = e^{-t}(c_1\cos 2t + c_2 \sin 2t)$

16

$y_p = Ae^{-2t}$

$y_p' = -2Ae^{-2t}$

$y_p'' = 4Ae^{-2t}$

$4Ae^{-2t} - 4Ae^{-2t} + 5Ae^{-2t} = e^{-2t}$

$5Ae^{-2t} = e^{-2t}$

$A = \frac{1}{5}$

$y_p = \frac{1}{5}e^{-2t}$

$$\boxed{y_c + y_p = e^{-t}(c_1\cos 2t + c_2 \sin 2t) + \frac{1}{5}e^{-2t}}$$

20

(12) 3. Find the general solution of the problem

$$y' + 2y = \cos(e^{2t}).$$

$$\mu(t) = \exp \int 2\, dt = e^{2t}$$

$$e^{2t}y' + 2e^{2t}y = \cos(e^{2t})$$

$$(e^{2t}y)' = \int \cos(e^{2t})\, dt$$

$$e^{2t}y = \cancel{\frac{\cos e^{2t}}{2}} + C$$

$$\boxed{y_p = \frac{(\cos e^{2t})e^{-2t}}{2} + Ce^{-2t}}$$

9

(14) 4. Solve the problem

$$y' = (1-2t)y^2, \quad y(0) = -\frac{1}{6}$$

and determine the interval on which the solution is valid.

$$\frac{dy}{dt} = (1-2t)y^2$$

$$\int \frac{1}{y^2}\, dy = \int (1-2t)\, dt$$

$$-\frac{1}{y} = t - t^2 + C$$

$$y = \frac{1}{t - t^2 + C}$$

$$\frac{1}{6} = \frac{1}{C} \Rightarrow C = 6$$

$$\boxed{y = \frac{1}{t - t^2 + 6} \qquad \cancel{t \neq 3}}$$

9

(20) 5. Consider the system

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}.$$

a. Classify the equilibrium point at the origin by shape and stability.

(0,0) is an ~~unstable spiral~~

$\lambda^2 - 4$ $\quad \lambda = \frac{0}{2} \pm \frac{\sqrt{4(-4)}}{2}$

$\lambda = \pm 2$ $\quad \cancel{\lambda = \pm 2i}$

For $\pm 2i$, we have a stable center.

$y = c_1 \cos 2t + c_2 \sin 2t$

b. Find straight-line solutions and use them to construct the general solution.

$$\begin{pmatrix} 0-\lambda & 1 \\ 4 & 0-\lambda \end{pmatrix} = \lambda^2 - 4$$

$$\lambda = \pm 2$$

if $\lambda = 2$

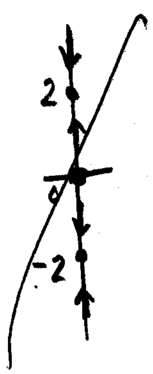
$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = -2v_1 + v_2 = 0 \quad v_2 = 2v_1 \implies \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

if $\lambda = -2$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 2v_1 + v_2 = 0 \quad 2v_1 = -v_2 \implies \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

✓ $$\boxed{y = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}}$$

c. Sketch the phase portrait.



12

(6) 6. Find the equilibrium points for the system

$$w' = p - 3w, \quad p' = p(1 - p - w).$$

$3w = p$ $\qquad$ $0 = 3w(1 - 3w - w)$

$0 = 3w(1 - 4w)$

$w = \frac{1}{4}$ or $w = 0$

6

if $w = \frac{1}{4}$ then $p = \frac{3}{4}$

if $w = 0$ the $p = 0$

$(0,0)$ and $(\frac{3}{4}, \frac{1}{4})$ order (p, w)

(10) 7. Sketch the phase line for the problem

$$y' = y^2 - 3y + 2$$

and determine the stability of equilibrium points.

$y^2 - 3y + 2$

$(y-1)(y-2)$

equillibrium pts are $y = 1$ and $y = 2$

10

2 is unstable

1 is stable

(12) 8. The radioactive isotope thorium-234 disintegrates at a rate proportional to the amount present. Write down a differential equation for the amount of thorium-234 present. If 100 mg of this material is reduced to 82 mg in one week, use the differential equation to determine a formula for the amount of thorium-234 present at any time.

$y(t) = y_0 e^{-kt}$

9

$82 = 100e^{-k(168)}$

$\ln\left(\frac{82}{100}\right) = -k(168)$

$.0012 = k$

$y(t) = y_0 e^{-.0012t}$