Notes on Surface Integrals

Surface integrals arise when we need to find the total of a quantity that is distributed on a surface. The standard integral with respect to area for functions of x and y is a special case, where the surface is given by $z = 0$. Other surfaces can lead to much more complicated integrals.

Types of Surface Integrals

Consider a surface S in three-dimensional space. Let $d\sigma$ represent the area of a little bit of the surface and let $\vec{\mathbf{n}}$ be the unit normal to the surface. If the surface is the boundary of a three-dimensional region, then $\vec{\mathbf{n}}$ is by definition the outward normal. If S is an open surface, then either of the two normal vectors can be used for $\vec{\mathbf{n}}$.

Let $f(\vec{\mathbf{r}})$ be a measure of some scalar quantity per unit area at the point indicated by the position vector $\vec{\mathbf{r}}$. Then $f\,d\sigma$ is the amount of that scalar quantity contained in the bit of surface, and the total amount of the scalar quantity is

$$\iint_S f\,d\sigma.$$

Now let $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ be a vector field, which we can think of as the velocity of a fluid in the region. Then $\vec{\mathbf{V}} \cdot \vec{\mathbf{n}}\,d\sigma$ is the flux of fluid through the bit of surface, and the total flux through the surface is

$$\iint_S \vec{\mathbf{V}} \cdot \vec{\mathbf{n}}\,d\sigma.$$

Evaluating Surface Integrals

For regions that lie in the xy plane, surface integrals of both types reduce to $\iint_R f\,dA$. We choose a two-dimensional coordinate system and write the region R, the function f and the differential dA in terms of the two coordinates. Integrals on other surfaces use the same idea, but with an important difference. When the surface is $z = 0$, we can use x and y to denote points on the surface. In other cases, we generally have to denote the three coordinates of surface points using just two parameters. Then we write the region S, the function f or $\vec{\mathbf{V}}$, and the differential $d\sigma$ or $\vec{\mathbf{n}}\,d\sigma$ in terms of the two parameters. Usually, the two parameters are chosen from the most convenient coordinate system.

There are four common special cases to consider: (1) S is parallel to a coordinate plane, such as the surface $z = 1$; (2) S is not parallel to a coordinate plane, but can be projected onto a coordinate plane[1]; (3) S is a portion of a circular cylinder; and (4) S is a portion of a sphere. Only rarely does one see a surface that does not fit one of these categories. Our goal for each case is to identify the appropriate parameters and the formulas for the differentials.

Case 1 — S is parallel to a coordinate plane

For surfaces $z = c$, we can just substitute c in for z and think of the integral as over the region R that sits below or above S in the xy plane. Similarly, we can use a region R in the yz plane for a surface $x = c$ and a region R in the zx plane for a surface $y = c$. We want to use the zx plane rather than the xz plane to be consistent with the right-hand rule.

Case 2a — S projects onto the xy plane

Each coordinate pair (x, y) corresponds to at most one point on the surface. We can write the scalar function f as $f(x, y, z(x, y))$, and R is the projection of S. Given the surface equation in the

[1] By this, we mean that the process of squashing the surface onto the coordinate plane does not cause points on the surface to merge with each other, as would happen with a sphere. Algebraically, we can project a surface onto a coordinate plane if we can solve the equation of the surface uniquely for the third variable.

form $g(x,y,z)=c$, and taking $\vec{\mathbf{n}}$ to be the normal with positive $\vec{\mathbf{k}}$ component, the formulas for the differentials are

$$d\sigma = \frac{|\vec{\nabla} g|}{|\vec{\nabla} g \cdot \vec{\mathbf{k}}|}\, dx\, dy, \qquad \vec{\mathbf{n}}\, d\sigma = \frac{\vec{\nabla} g}{\vec{\nabla} g \cdot \vec{\mathbf{k}}}\, dx\, dy.$$

Case 2b — S is of the form $z = f(r)$

Here we can project onto the xy plane, but then we want to use polar coordinates. The necessary formulas are

$$d\sigma = \sqrt{[f'(r)]^2+1}\; r\, dr\, d\theta, \qquad \vec{\mathbf{n}}\, d\sigma = \langle -f'(r)\cos\theta, -f'(r)\sin\theta, 1\rangle\; r\, dr\, d\theta.$$

Cases 2c,d — S projects onto the yz or zx plane

These cases are analogous to Case 2a. Taking $\vec{\mathbf{n}}$ to be the normal with positive $\vec{\mathbf{i}}$ (for yz) or $\vec{\mathbf{j}}$ (for zx) component, the formulas for the differentials are

$$d\sigma = \frac{|\vec{\nabla} g|}{|\vec{\nabla} g \cdot \vec{\mathbf{i}}|}\, dy\, dz, \qquad \vec{\mathbf{n}}\, d\sigma = \frac{\vec{\nabla} g}{\vec{\nabla} g \cdot \vec{\mathbf{i}}}\, dy\, dz.$$

for projection onto the yz plane and

$$d\sigma = \frac{|\vec{\nabla} g|}{|\vec{\nabla} g \cdot \vec{\mathbf{j}}|}\, dx\, dz, \qquad \vec{\mathbf{n}}\, d\sigma = \frac{\vec{\nabla} g}{\vec{\nabla} g \cdot \vec{\mathbf{j}}}\, dx\, dz.$$

for projection onto the zx plane.

Case 3 — S is a portion of a circular cylinder $r =$ const

For a surface $r = c > 0$, we can use θ and z as the parameters. The vector $\langle x, y, 0\rangle$ is an outward normal, so we can use it for $\vec{\mathbf{n}}$. The necessary formulas are

$$d\sigma = r\, dz\, d\theta, \qquad \vec{\mathbf{n}} = \frac{\langle x,y,0\rangle}{r} = \langle \cos\theta, \sin\theta, 0\rangle.$$

The flux integral, for example, is then

$$\iint_S \vec{\mathbf{V}} \cdot \vec{\mathbf{n}}\, d\sigma = \int_{\theta_1}^{\theta_2} \int_{z_1}^{z_2} \vec{\mathbf{V}}(r\cos\theta, r\sin\theta, z) \cdot \langle \cos\theta, \sin\theta, 0\rangle\; r\, dz\, d\theta.$$

Case 4 — S is a portion of a sphere $\rho =$ const

For a surface $\rho = c > 0$, we can use θ and ϕ as the parameters. The position vector $\vec{\mathbf{r}}$ is an outward normal, so we can use it for $\vec{\mathbf{n}}$. The necessary formulas are

$$d\sigma = \rho^2 \sin\phi\, d\phi\, d\theta, \qquad \vec{\mathbf{n}} = \frac{\vec{\mathbf{r}}}{\rho}.$$

The scalar surface integral is then

$$\iint_S f\, d\sigma = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} f(\rho,\theta,\phi)\; \rho^2 \sin\phi\, d\phi\, d\theta,$$

and the flux integral is

$$\iint_S \vec{\mathbf{V}} \cdot \vec{\mathbf{n}}\, d\sigma = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} (\vec{\mathbf{V}} \cdot \vec{\mathbf{r}})(\theta,\phi)\; \rho \sin\phi\, d\phi\, d\theta.$$

(Note that the differential for the flux integral has only one factor of ρ.) This form is especially convenient when one can determine the angle α between $\vec{\mathbf{V}}$ and $\vec{\mathbf{r}}$ and use the formula

$$\vec{\mathbf{V}} \cdot \vec{\mathbf{r}} = |\vec{\mathbf{V}}||\vec{\mathbf{r}}| \cos\alpha.$$