

Notes on Surface Integrals

Surface integrals arise when we need to find the total of a quantity that is distributed on a surface. The standard integral with respect to area for functions of x and y is a special case, where the surface is given by $z = 0$. Other surfaces can lead to much more complicated integrals.

Types of Surface Integrals

Consider a surface S in three-dimensional space. Let $d\sigma$ represent the area of a little bit of the surface and let $\vec{\mathbf{n}}$ be the unit normal to the surface. If the surface is the boundary of a three-dimensional region, then $\vec{\mathbf{n}}$ is by definition the outward normal. If S is an open surface, then either of the two normal vectors can be used for $\vec{\mathbf{n}}$.

Let $f(\vec{\mathbf{r}})$ be a measure of some scalar quantity per unit area at the point indicated by the position vector $\vec{\mathbf{r}}$. Then $f d\sigma$ is the amount of that scalar quantity contained in the bit of surface, and the total amount of the scalar quantity is

$$\iint_S f d\sigma.$$

Now let $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ be a vector field, which we can think of as the velocity of a fluid in the region. Then $\vec{\mathbf{V}} \cdot \vec{\mathbf{n}} d\sigma$ is the flux of fluid through the bit of surface, and the total flux through the surface is

$$\iint_S \vec{\mathbf{V}} \cdot \vec{\mathbf{n}} d\sigma.$$

Evaluating Surface Integrals

For regions that lie in the xy plane, surface integrals of both types reduce to $\iint_R f dA$. We choose a two-dimensional coordinate system and write the region R , the function f and the differential dA in terms of the two coordinates. Integrals on other surfaces use the same idea, but with an important difference. When the surface is $z = 0$, we can use x and y to denote points on the surface. In other cases, we generally have to denote the three coordinates of surface points using just two parameters. Then we write the region S , the function f or $\vec{\mathbf{V}}$, and the differential $d\sigma$ or $\vec{\mathbf{n}} d\sigma$ in terms of the two parameters. Usually, the two parameters are chosen from the most convenient coordinate system.

There are four common special cases to consider: (1) S is parallel to a coordinate plane, such as the surface $z = 1$; (2) S is not parallel to a coordinate plane, but can be projected onto a coordinate plane¹; (3) S is a portion of a circular cylinder; and (4) S is a portion of a sphere. Only rarely does one see a surface that does not fit one of these categories. Our goal for each case is to identify the appropriate parameters and the formulas for the differentials.

Case 1 — S is parallel to a coordinate plane

For surfaces $z = c$, we can just substitute c in for z and think of the integral as over the region R that sits below or above S in the xy plane. Similarly, we can use a region R in the yz plane for a surface $x = c$ and a region R in the zx plane for a surface $y = c$. We want to use the zx plane rather than the xz plane to be consistent with the right-hand rule.

Case 2a — S projects onto the xy plane

Each coordinate pair (x, y) corresponds to at most one point on the surface. We can write the scalar function f as $f(x, y, z(x, y))$, and R is the projection of S . Given the surface equation in the

¹By this, we mean that the process of squashing the surface onto the coordinate plane does not cause points on the surface to merge with each other, as would happen with a sphere. Algebraically, we can project a surface onto a coordinate plane if we can solve the equation of the surface uniquely for the third variable.

form $g(x, y, z) = c$, and taking $\vec{\mathbf{n}}$ to be the normal with positive $\vec{\mathbf{k}}$ component, the formulas for the differentials are

$$d\sigma = \frac{|\vec{\nabla}g|}{|\vec{\nabla}g \cdot \vec{\mathbf{k}}|} dx dy, \quad \vec{\mathbf{n}} d\sigma = \frac{\vec{\nabla}g}{\vec{\nabla}g \cdot \vec{\mathbf{k}}} dx dy.$$

Case 2b — S is of the form $z = f(r)$

Here we can project onto the xy plane, but then we want to use polar coordinates. The necessary formulas are

$$d\sigma = \sqrt{[f'(r)]^2 + 1} r dr d\theta, \quad \vec{\mathbf{n}} d\sigma = \langle -f'(r) \cos \theta, -f'(r) \sin \theta, 1 \rangle r dr d\theta.$$

Cases 2c,d — S projects onto the yz or zx plane

These cases are analogous to Case 2a. Taking $\vec{\mathbf{n}}$ to be the normal with positive $\vec{\mathbf{i}}$ (for yz) or $\vec{\mathbf{j}}$ (for zx) component, the formulas for the differentials are

$$d\sigma = \frac{|\vec{\nabla}g|}{|\vec{\nabla}g \cdot \vec{\mathbf{i}}|} dy dz, \quad \vec{\mathbf{n}} d\sigma = \frac{\vec{\nabla}g}{\vec{\nabla}g \cdot \vec{\mathbf{i}}} dy dz.$$

for projection onto the yz plane and

$$d\sigma = \frac{|\vec{\nabla}g|}{|\vec{\nabla}g \cdot \vec{\mathbf{j}}|} dx dz, \quad \vec{\mathbf{n}} d\sigma = \frac{\vec{\nabla}g}{\vec{\nabla}g \cdot \vec{\mathbf{j}}} dx dz.$$

for projection onto the zx plane.

Case 3 — S is a portion of a circular cylinder $r = \text{const}$

For a surface $r = c > 0$, we can use θ and z as the parameters. The vector $\langle x, y, 0 \rangle$ is an outward normal, so we can use it for $\vec{\mathbf{n}}$. The necessary formulas are

$$d\sigma = r dz d\theta, \quad \vec{\mathbf{n}} = \frac{\langle x, y, 0 \rangle}{r} = \langle \cos \theta, \sin \theta, 0 \rangle.$$

The flux integral, for example, is then

$$\iint_S \vec{\mathbf{V}} \cdot \vec{\mathbf{n}} d\sigma = \int_{\theta_1}^{\theta_2} \int_{z_1}^{z_2} \vec{\mathbf{V}}(r \cos \theta, r \sin \theta, z) \cdot \langle \cos \theta, \sin \theta, 0 \rangle r dz d\theta.$$

Case 4 — S is a portion of a sphere $\rho = \text{const}$

For a surface $\rho = c > 0$, we can use θ and ϕ as the parameters. The position vector $\vec{\mathbf{r}}$ is an outward normal, so we can use it for $\vec{\mathbf{n}}$. The necessary formulas are

$$d\sigma = \rho^2 \sin \phi d\phi d\theta, \quad \vec{\mathbf{n}} = \frac{\vec{\mathbf{r}}}{\rho}.$$

The scalar surface integral is then

$$\iint_S f d\sigma = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} f(\rho, \theta, \phi) \rho^2 \sin \phi d\phi d\theta,$$

and the flux integral is

$$\iint_S \vec{\mathbf{V}} \cdot \vec{\mathbf{n}} d\sigma = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} (\vec{\mathbf{V}} \cdot \vec{\mathbf{r}})(\theta, \phi) \rho \sin \phi d\phi d\theta.$$

(Note that the differential for the flux integral has only one factor of ρ .) This form is especially convenient when one can determine the angle α between $\vec{\mathbf{V}}$ and $\vec{\mathbf{r}}$ and use the formula

$$\vec{\mathbf{V}} \cdot \vec{\mathbf{r}} = |\vec{\mathbf{V}}| |\vec{\mathbf{r}}| \cos \alpha.$$