Optimization

The material on optimization in a calculus book is organized by method. This is good for learning the various techniques, but it does not give a framework for use in solving problems that are not sorted by type. Here we present such a framework.

Let $f(x, y)$ be a function of two variables defined on some region in the xy-plane, possibly the whole xy-plane. There are two distinct types of extremum problems—local problems and global problems. We consider these in turn.

Local Extrema

Local extrema are points that have the largest or smallest function value relative to their immediate neighbors. Because they are local phenomena, they are found using methods of calculus.

Local Extrema in a Plane Region Usually we want to find local extrema in a 2-dimensional region. A point P in 2D is a local maximum, for example, if it is possible to find a small circle around the point P on which the function value is never larger than f at P.

1. Local extrema in 2D can occur only at critical points: points where $\vec{\nabla} f = \vec{0}$ or where the gradient is undefined.

2. Let $D(x, y) = f_{xx}f_{yy} - (f_{xy})^2$. If $D > 0$ at a critical point, then the critical point is a local extremum. The signs of f_{xx} and f_{yy} make it clear whether the point is a maximum or a minimum.

3. If $D < 0$ at a critical point, then the critical point is a saddle point.

4. If $D = 0$ at a critical point, or if the gradient is undefined at the critical point, then it is much more difficult to test for extrema. Sometimes it helps to sketch a contour plot near the point.

Local Extrema on a Curve in the Plane Sometimes we want to find local extrema on a curve rather than a 2-dimensional region. If a point P is a local maximum on a curve, it means only that the function value of neighboring points on that curve are no larger than f at P. Points that are local extrema on a curve are generally not local extrema in a plane region.

5. Any point that is a local extremum in a plane region is also a local extremum on a curve in the plane.

6. Local extrema on a curve can be found by 1-variable methods if one can solve the equation of the curve for either x or y and substitute the result into f to get a function of 1 variable. For example, local extrema on the curve $y = 1$ can be found by examining the function $g(x) = f(x, 1)$ and local extrema on the curve $y = x^2$ can be found from $g(x) = f(x, x^2)$.

7. Local extrema on a curve can only occur at points where the Lagrange multiplier rule is satisfied. If the curve is given by the equation $g(x, y) = c$, then the points must satisfy $\vec{\nabla} f = \lambda \vec{\nabla} g$ for some real number λ. The vector equation and the equation of the curve together make a set of three equations for the unknowns x, y, and λ.

8. There is no simple calculation test to determine whether a point satisfying the Lagrange multiplier rule is a local maximum, a local minimum, or an inflection point. The best way to tell is to examine the contour plot of the function.

Global Extrema

Global extrema are points that have the largest or smallest function value relative to all points in the region of interest. Because they are not local phenomena, methods of calculus are generally not sufficient to determine them.

Global Extrema on a Closed Bounded Region

9. A continuous function defined on a closed bounded region always has a global maximum and a global minimum.

10. Only certain points within the closed bounded region can be considered as candidates for global extrema. Candidates include any critical points in the region, any boundary points that are local extrema on the boundary curve, and any corner points where two different boundary curves intersect. If there are only two candidates, then one must be the global max and the other the global min. If there are more candidates, the surest way to tell which are the global max and min is to compare the function values at the points.

Global Extrema on an Open or Unbounded Region

11. With a discontinuous function, an open region, or an unbounded region, there is no guarantee of global extrema. A typical case is that of finding global extrema over the whole xy-plane. In order for there to be a global maximum, the function must not go off to infinity along any path in the xy-plane.

12. In order for a point to be a global extremum on an open or unbounded region, it must at least be a local extremum.

13. Suppose P is the local maximum for which the function value is the largest. Then either P is the global maximum or there is none. If there is no global maximum, it is because the function increases in a direction that goes off to infinity or approaches an open boundary. There are some general cases for which a clear result can be given. Suppose the region is the xy-plane and D is continuous and positive everywhere. In this general setting, P is necessarily the global maximum.[1] The same result holds if the region is convex.[2] If no general result can be applied to the specific problem at hand, then one can look at cross sections and contour plots to try to determine if the largest local maximum is a global maximum.

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[1] Since P is a local maximum and D continuous and positive, we have $f_{xx}, f_{yy} < 0$ in the region. Let P be (x_0, y_0) and choose any point (x_0+u, y_0+v). Let $F(s) = f(x_0+su, y_0+sv)$. Then $F'' = u^2 f_{xx} + 2uv f_{xy} + v^2 f_{yy} = [(uf_{xx} + vf_{xy})^2 + v^2 D]/f_{xx} \leq 0$ on the line connecting the points, so $f(x_0+u, y_0+v) \leq f(x_0, y_0)$.

[2] Choose two points in the region and connect them with a straight line. If the points can be chosen so that the line is not wholly in the region, then the region is not convex.