Line Integral Summary

We assume that the vector field $\vec{\mathbf{F}}$ and curve C are given in terms of x and y. Everything given here generalizes to 3D if done correctly. Green's Theorem (the final evaluation technique) requires special care; it generalizes only to curves in a different coordinate plane in which the coordinates are ordered by the right-hand rule and the curl is appropriately defined.

Notation

vector field	$\vec{\mathbf{F}} = F_1(x,y)\vec{\mathbf{i}} + F_2(x,y)\vec{\mathbf{j}} \equiv \langle F_1(x,y), F_2(x,y)\rangle \equiv (F_1(x,y), F_2(x,y))$
oriented curve C	$\vec{\mathbf{r}} = x(t)\vec{\mathbf{i}} + y(t)\vec{\mathbf{j}} \equiv (x(t), y(t))$
line integral	$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \equiv \int_C F_1\,dx + F_2\,dy \equiv \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds$

The notation $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds$ is becoming less common. We include it here for reference. $\vec{\mathbf{T}}(x,y)$ denotes a unit vector tangent to the curve at a point (x,y) on C.

Line integrals are sometimes written with the symbol $\oint$ to indicate that the curve C must be closed.

Special Vector Fields

In this list, assume that the vector field $\vec{\mathbf{F}}$ has continuous first partial derivatives on a region containing the curve C. If C is closed, then the region also contains the interior of C. With these assumptions, the following definitions are equivalent:

gradient field	$\vec{\mathbf{F}} = \vec{\nabla} f$ for some $f(x,y)$
path-independent field	$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ depends only on the endpoints of C.
conservative field	$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$ for **all** closed curves C.
irrotational field	$\vec{\text{curl}}\, \vec{\mathbf{F}} = \vec{\mathbf{0}}$, where curl $\vec{\mathbf{F}} = \langle 0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\rangle$.

Note that the first three definitions do not require the vector field to have continuous first derivatives, although they do require the vector field to be defined at all points in the region. However, the equivalence of the properties is not guaranteed when it does not have continuous first derivatives. In particular, watch out for vector fields that are undefined at one point.

We can determine if $\vec{\mathbf{F}}$ satisfies these properties by trying to solve $\vec{\mathbf{F}} = \vec{\nabla} f$ for f or by checking the curl. We obviously can't check path-independence and conservation independently.

Evaluating Line Integrals

1. We can always use a parameterization to reduce a line integral to a single variable integral.
$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b (\vec{\mathbf{F}} \cdot \vec{\mathbf{r}}')\,dt,$$
where the derivative is with respect to the parameter, the integrand is written entirely in terms of the parameter, and $a \le t \le b$.

2. If the vector field is path-independent, then one can replace the curve C with a different curve having the same beginning and ending point.

3. If the vector field is a gradient field and the curve runs from P to Q, then $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f|_Q - f|_P$.

4. If the vector field has continuous first derivatives on the closed curve C and its interior R, then
$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

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