Directional Slope

The equation of a linear function $f(x,y)$ can be written as

$$f(x,y) = c + mx + ny,$$

where c, m, and n are constants.

Significance of m and n Consider two points, (x,y) and $(x+\Delta x, y+\Delta y)$. Let $\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y)$. We may then calculate the change in f as

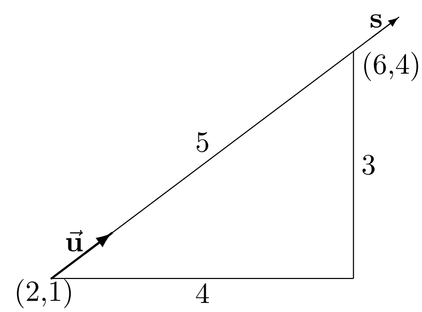
$$\Delta f = [c + m(x+\Delta x) + n(y+\Delta y)] - [c + mx + ny] = m\Delta x + n\Delta y.$$

This formula shows us the significance of the parameters m and n. Suppose we consider two points having the same y coordinate. Then $\Delta y = 0$. We can rearrange the last formula to get

$$m = \frac{\Delta f}{\Delta x}.$$

Thus, m is the slope of the function in the x-direction. Similarly, n is the slope of the functionin the y-direction. Another way to state this is that m is the slope of the function in the direction given by the unit vector $\vec{i}$ and n is the slope of the function in the direction given by the unit vector $\vec{j}$.

Slope in the direction of a unit vector $\vec{u}$ Consider a specific example. We are interested in knowing the slope at the point (2,1) in the direction of the point (6,4).



Note that we have two different labels on the ray that begins at (2,1) and passes through the point (6,4). There is a coordinate label s and a unit vector label $\vec{u}$. This is analogous to the standard coordinate plane, in which the axes can be labeled with the coordinates x and y and also the unit vectors $\vec{i}$ and $\vec{j}$.

The slope of f in the horizontal direction can be thought of as the rate of change of f with respect to x. Similarly, the slope of f along the path from (2,1) to (6,4) can be thought of as the rate of change of f with respect to s, where s is a coordinate that goes in the given direction. The slope in the s direction is $\Delta f/\Delta s$. Here, the change in f is $m\Delta x + n\Delta y = 4m + 3n$. The change in s is the distance between the points, or 5. So the slope is

$$\text{slope } = \frac{\Delta f}{\Delta s} = \frac{4m+3n}{5} = 0.8m + 0.6n.$$

The slope of f in the horizontal direction can also be thought of as the slope of f in the $\vec{i}$ direction. Similarly, the slope of f along the path from (2,1) to (6,4) can be thought of as the slope of f in the $\vec{u}$ direction. The unit vector $\vec{u}$ is obtained by dividing the displacement vector $4\vec{i} + 3\vec{j}$ by its length 5. Thus,

$$\vec{u} = 0.8\vec{i} + 0.6\vec{j}.$$

Now we define the **gradient vector**, which we will call g for now, as

$$\vec{g} = m\vec{i} + n\vec{j}.$$

The components of the gradient vector are the slopes in the corresponding directions. Is there a way to calculate the slope of f in the $\vec{u}$ direction using the gradient and direction vectors? Clearly the answer is that the slope is the dot product of the two vectors:

$$\text{slope } = \vec{g} \cdot \vec{u}.$$

Summary

There are two ways to think of the slope of a linear function in a given direction.

- We can use a symbol, such as s, to represent distance in the given direction. Then we compute the slope in the given direction as the change in f divided by the change in s.

$$\text{slope } = \frac{\Delta f}{\Delta s}.$$

- We can describe the direction using the unit vector $\vec{u}$ that points in the given direction. We can define the gradient vector to be the vector in which each component is the slope in the corresponding direction; thus, the $\vec{i}$ component is the slope in the $\vec{i}$ direction and so on. Then we compute the slope in the given direction as the dot product of the gradient and direction vectors.

$$\text{slope } = \vec{g} \cdot \vec{u}.$$

In 1-variable calculus, we use the concept of the slope of a line to define the derivative, which represents the slope of a curve. Similarly, in multi-variable calculus, we will use the concept of the slope of a plane to define the directional derivative, which represents the slope of a surface. The material here on slope of a linear function thus provides the precalculus foundation for the derivative in multi-dimensional calculus.

Exercises

1. Let $f = 2 + 3x + 2y$. Use both definitions of the slope to calculate the slope of f in the direction of the vector $2\vec{i} - \vec{j}$.

2. Let $f = 5 - x + 2y$. Use both definitions of the slope to calculate the slope of f in the direction of the vector $\vec{i} + 3\vec{j}$.

3. Let $f = 3 - 2x - 4y$. Use both definitions of the slope to calculate the slope of f in the direction of the vector $-2\vec{i} - 3\vec{j}$.

4. Let f be defined by the contour diagram in Figure 11.73. Determine the slope of f in the direction indicated by the ray from the point (-1,-1) through the point (1,2).