The Differential

The **differential** of a function $f(x, y, z)$ (similarly for functions of two variables) is an expression that uses the partial derivatives of the function:

$$df = f_x\,dx + f_y\,dy + f_z\,dz \tag{1}$$

It has some applications, for example, as a convenient way of applying the chain rule in some examples, and it is used for some theoretical developments that don't usually appear in a multivariable calculus course.

Misuse of the Differential

Most mathematics books also use the differential for linear approximation by using finite values for dx, dy, and dz and then getting a finite approximation for df. This is a very unfortunate use of the differential because it prevents us from being able to reserve use of the differential in the purpose for which it was intended–as notation to indicate arbitrarily small changes.

Differentials appear in integrals; for example, we will eventually write an integral for the volume of a region in space as

$$V = \iint_R f(x, y)\,dA,$$

where R is the (2D) projection of the solid onto the xy-plane, $f(x, y)$ is the height of the solid over the point (x, y), and dA is a differential in 2D. The usual way integrals are presented is as limits of Riemann sums, with the differential as part of the notation but having no independent meaning. It is much easier to do applications with definite integrals if we interpret dA as an infinitesimal bit af area in the xy-plane located at the point (x, y). With this interpretation, we can identify $f(x, y)\,dA$ as the infinitesimal bit of volume located over the point (x, y) and interpret the integral as an infinite sum of these infinitesimal volumes.

If we use Equation (1) for finite linear approximation, say with $dx = 0.1$, then we are employing the notation of the differential in a way that is inconsistent with its use in definite integrals. Inconsistent use of notation in mathematics makes it much harder for students of mathematics to learn the concepts. Mathematicians should be more careful not to do this. In my class, we will NEVER use the differential for anything that is large enough to be assigned a numerical value, no matter how small the value. It will ALWAYS refer only to quantities that are arbitrarily small, meaning that they are individually 0 but collectively finite when infinitely many of them are added together.

The Right Way to Do Linear Approximation

The notation of the differential contributes nothing to linear approximation. We can simply use finite differences, being careful to use "approximately equal" rather than "equal." For example, to calculate the change in a function $f(x, y) = x^2y + y^2$ when moving away from the point $(2, 1)$, we calculate the partial derivatives,

$$f_x = 2xy, \qquad f_y = x^2 + 2y,$$

evaluate them at the point,

$$f_x(2, 1) = 4, \qquad f_y(2, 1) = 6,$$

and write the linear approximation formula

$$\Delta f \approx 4\Delta x + 6\Delta y$$

using the values of the partial derivatives. In a specific instance, we are given finite values of Δx and Δy and use them to calculate an approximation for Δf. For example, at the point $(2.1, 0.8)$, we have

$$\Delta x = 0.1, \qquad \Delta y = -0.2, \qquad \Delta f \approx 4(0.1) + 6(-0.2) = -0.8,$$

so

$$f(2.1, 0.8) \approx f(2, 1) + \Delta f = 4.2.$$