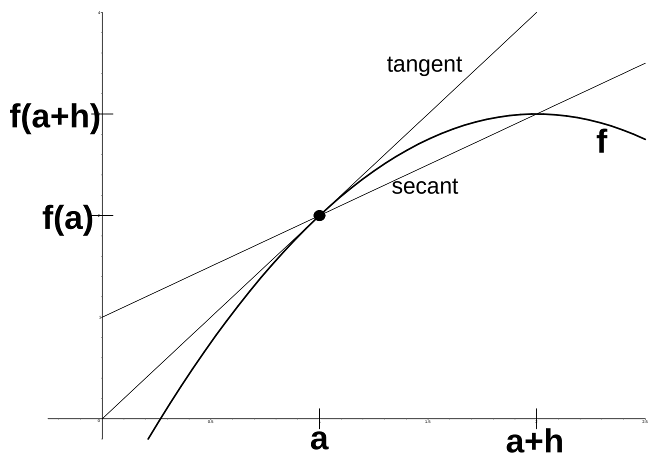
The Concept of the Derivative

The derivative of a nonlinear function is related to the rate of change of a linear function, which is the same thing as the slope of a line. We consider the problem of calculating the slope of the tangent line to a curve, and then we use the solution to define the derivative. The discussion refers to the figure below.



The Slope of a Secant Suppose we have a function $f(t)$, and we want to determine the slope of the tangent line to the function f at a point $t = a$. This poses a problem because algebraic methods can be used to calculate the slope of a line only when two points on the line are known. Only one point, namely the point $(a, f(a))$, on the tangent line is known.

A secant line to a curve is a line that connects two points on the graph. Since two points on a secant line are known, we can always calculate the slope of a secant line. Let's consider the secant line that appears in the figure. This line connects the points $(a, f(a))$ and $(a+h, f(a+h))$. Using the standard calculation for the slope of a line, we determine the slope of the secant line to be

$$m_{\text{sec}} = \frac{\Delta f}{\Delta t} = \frac{f(a+h) - f(a)}{h}.$$

The Slope of the Tangent It is clear from the figure that the slope of a secant is different from the slope of the tangent. However, it is also clear that a secant slope approximation can always be improved by choosing a smaller value of h. Notice that the formula for the secant slope works for any non-zero value of h, no matter how small. However, it does not work for the value $h = 0$. This means that we cannot directly calculate the tangent slope; however, we can indirectly calculate it by determining what will happen to the secant slope as h becomes arbitrarily close to 0. This can sometimes be done exactly, but it can always be done approximately, simply by selecting a very small vale of h. Mathematically, the concept is written using the limit notation.

$$m_{\text{tan}} = \lim_{h \to 0} m_{\text{sec}}.$$

The Definition of the Derivative The terminology of the tangent slope and the notation m_{tan} are very clumsy and refer neither to the specific function whose tangent slope is being determined nor to the point at which the slope is being determined. A better terminology is that of the **derivative**; specifically, the derivative of f at a is defined to be the slope of the tangent to the curve $f(t)$ at the point $t = a$, and is denoted $f'(a)$.

$$f'(a) = m_{\text{tan}} = \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The Derivative as a Function Notice that there is nothing in the development above that restricts the choice of the point a. Since a is arbitrary and the definition of $f'(a)$ does not use the symbol t, there is no loss of generality in rewriting the definition so that the derivative is a function of t, rather than merely the slope at a particular point.

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.$$

Although we shall stop using the notation m_{tan}, it is important to remember what $f'(t)$ is: it is the function that calculates the slope of the curve $f(t)$ at all points t that have a tangent line. Nearly all applications of the derivative can be thought of in terms of this graphical interpretation; for example, velocity $v(t)$ is the slope of the graph of the position function $s(t)$. Thus, $v(t) = s'(t)$.