A Cohen-Macaulay Property for Non-Noetherian Rings

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Abstract

Have you ever wondered what Cohen-Macaulay should mean for non-Noetherian rings? Apparently Glaz did, and Hamilton/Marley decided to answer the question. More specifically, they gave a definition of Cohen-Macaulay that ensures that all (not necessarily Noetherian) regular rings are Cohen-Macaulay. There are some obvious difficulties to deal with, and the Cohen-Macaulay property isn’t quite so well-behaved in the non-Noetherian setting. We shall delve into some of these issues.

Based on the paper Non-Noetherian Cohen-Macaulay Rings by Tracy Dawn Hamilton and Thomas Marley. Throughout, $R$ is commutative with 1.

1 What do we want?

There are numerous characterizations of Noetherian Cohen-Macaulay rings, so the first step in seeking to generalize the Cohen-Macaulay property is to decide either which characterization is the most useful to us, or what properties we want Cohen-Macaulay rings to possess. Glaz originally asked for a characterization of Cohen-Macaulay rings that had the property that all regular rings are Cohen-Macaulay. The answer is yes, at least for coherent regular rings.

The definition we (alright, they) choose to generalize is this: A Noetherian ring $R$ is Cohen-Macaulay if and only if any sequence $x_1, \ldots, x_n$ of elements in $R$ generating a height $n$ ideal is a regular sequence (up to some permutation, in the global case). Height is poorly behaved in non-Noetherian rings, so we cannot just drop the Noetherian condition. Indeed, we need to replace the height condition with something that works without the Noetherian property.

The plan, broadly, is: Replace the height condition on $\underline{x} = x_1, \ldots, x_n$ with conditions on Cech cohomology $H^n(\underline{x}; R)$ and Koszul homology $H^i(\underline{x}; R)$. But first, preliminaries:
2 What definition do we pick?

**Definition 2.1.** Let $R$ be a commutative ring, $\overline{x} = x_1, \ldots, x_n$ a finite sequence in $R$. Let $I = (\overline{x})$. The following are equivalent:

1. The sequence $\overline{x}$ is weakly pro-regular.
2. $H^i_I(E) = 0$ for all injective $R$-modules $E$ and $i \neq 0$.
3. For every $R$-modules $M$, there exist a natural isomorphism
   \[ H^i_I(M) \to H^1_I(M) \]
   where $H^i_I(M)$ is the $i$-th local cohomology of $M$ with support in $V(I)$.

**Note 2.1.** In a Noetherian ring, the last condition is true for any finite sequence; hence any finite sequence in a Noetherian ring is weakly pro-regular.

**Definition 2.2.** Let $R$ be commutative, $\overline{x} = x_1, \ldots, x_n$ a sequence in $R$. We call $\overline{x}$ a parameter sequence if

1. The sequence $\overline{x}$ is weakly pro-regular,
2. $(\overline{x})R \neq R$, and
3. For every prime $p$ containing $(\overline{x})R$, we have $H^n_p(R) \neq 0$.

We say that $\overline{x}$ is a strong parameter sequence on $R$ if $x_1, \ldots, x_i$ is a parameter sequence for all $i \leq n$.

**Remark 2.2.** Notice that if $R$ is Noetherian, we have that $\overline{x}$ is a parameter sequence on $R$ if and only if $\text{ht}(\overline{x}) = n$.

This immediately suggests:

**Definition 2.3.** A ring $R$ is called Cohen-Macualay if every strong parameter sequence on $R$ is a regular sequence.

3 What do we get?

For Noetherian rings, we usually talk about regular sequences using the notion of grade and its relation to dimension, i.e., locally, depth equals dimension. But grade behaves poorly in non-Noetherian rings. In effect, we lose: $\text{grade}(I, M) > 0$ if and only if $(0 :_M I) = 0$. However, this disappears in polynomial extensions of rings, which leads to the notion of polynomial grade.

**Definition 3.1.** Let $I$ be an ideal of a ring $R$, $M$ an $R$-module. The polynomial grade of $I$ on $M$ is defined by

\[ p\text{-grade}(I, M) = \lim_{m \to \infty} \text{grade}(IR[t_1, \ldots, t_m], R[t_1, \ldots, t_m] \otimes_R M). \]
This is useful because:

**Lemma 3.1.** Let $R$ be a ring, $I = (x_1, \ldots, x_n)$ and $M$ an $R$-module. Then $\text{grade}(IR[t], R[t] \otimes_R M) > 0$ if and only if $(0:_{M} I) = 0$. In particular, $(0:_{M} I) = 0$ if and only if $x_1 + x_2 t + \cdots + x_n t^{n-1} \in IR[t]$ is a non-zero divisor on $R[t] \otimes_R M$.

**Proof.** See: Northcott, D.G. Finite Free Resolutions

**Theorem 3.2.** Let $R$ be a ring. Then the following are equivalent:

1. $R$ is Cohen-Macaulay.
2. For every strong parameter sequence $x_1, \ldots, x_n$, we have $\text{grade}(\langle x \rangle) = n$,
3. For every strong parameter sequence $x_1, \ldots, x_n$, we have $\text{p-grade}(\langle x \rangle) = n$,
4. For every strong parameter sequence $x$, we have $H_i(\langle x \rangle; R) = 0$ for all $i \geq 1$.
5. For every strong parameter sequence $x$ of length $n$, we have $H_i(\langle x \rangle; R) = 0$ for all $i < n$.

Note: Unlike the Noetherian case, it is not sufficient for 3 to hold merely for all maximal strong parameter sequences.

**Proposition 3.3.** Let $R$ be a ring.

1. If $\dim(R) = 0$, $R$ is Cohen-Macaulay.
2. If $R$ is a 1 dimensional domain, $R$ is Cohen-Macaulay.

Indeed, if $R$ is zero-dimensional, there are no parameter sequences. If it is a one-dimensional domain, every parameter sequence has length 0 or 1, and every non-zero element is a non-zero divisor.

Why is this a good definition of Cohen-Macaulay?

**Proposition 3.4.** Let $f: R \to S$ be a faithfully flat ring homomorphism. If $S$ is Cohen-Macaulay, then so is $R$.

**Corollary 3.5.** Let $R$ be a ring such that $R[t]$ is Cohen-Macaulay. Then $R$ is as well.

**Proposition 3.6.**

1. Let $R$ be a ring and suppose $R_m$ is Cohen-Macaulay for all maximal ideals $m$. Then $R$ is Cohen-Macaulay.
2. Let $R$ be coherent regular ring. $R$ is Cohen-Macaulay.
3. If every ideal generated by a strong parameter sequence is unmixed, then $R$ is Cohen-Macaulay.