Problems and Progress:  
A survey on fat points in $\mathbb{P}^2$

Brian Harbourne 

Department of Mathematics and Statistics  
University of Nebraska-Lincoln  
Lincoln, NE 68588-0323  
email: bharbour@math.unl.edu  
WEB: http://www.math.unl.edu/~bharbour/  

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Abstract: This paper, which expands on a talk given at the International Workshop on Fat Points, February 9-12, 2000, in Naples, Italy, surveys problems and progress on certain problems involving numerical characters for ideals $I(Z)$ defining fat points subschemes $Z = m_1p_1 + \cdots + m_np_n \subset \mathbb{P}^2$ for general points $p_i$. In addition to presenting some new results, a collection of MACAULAY 2 scripts for computing actual or conjectured values of (or bounds on) these characters is included. One such script, findres, for example, computes the syzygy modules in a minimal free resolution of the ideal $I(Z)$ for any such $Z$ with $n \leq 8$; since findres does not rely on a Gröbner basis calculation, it is much faster than routines that do.

I. Introduction

This paper surveys work on certain problems involving fat points subschemes of $\mathbb{P}^2$. To encourage experimentation, I have included a number of MACAULAY 2 scripts for doing explicit calculations. To simplify using them, I’ve included them in the TeXfile for this paper in a verbatim listing, without any intervening TeX control sequences. Thus if you have (or obtain, from, say, http://www.math.unl.edu/~bharbour/Survey.tex) the TeXlisting for this paper, you can simply copy the lines for the necessary MACAULAY 2 scripts from this paper directly into MACAULAY, without any additional editing.

Although the most general definition of a fat points subscheme involves the notion of infinitely near points (see [H6]), it is simpler here to define a fat points subscheme of $\mathbb{P}^2$ to be a subscheme $Z$ defined by a homogeneous ideal $I \subset R$ of the form $I(p_1)^{m_1} \cap \cdots \cap I(p_n)^{m_n}$, where $p_1, \ldots, p_n$ are distinct points of $\mathbb{P}^2$, $m_1, \ldots, m_n$ are nonnegative integers and $R = k[\mathbb{P}^2]$ is the homogeneous coordinate ring of $\mathbb{P}^2$ (i.e., a polynomial ring in 3

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variables, $x$, $y$ and $z$, over an algebraically closed field $k$). It is convenient to denote $Z$ by $Z = m_1p_1 + \cdots + m_np_n$ and to denote $I$ by $I(Z)$.

For another perspective, a homogeneous polynomial $f \in R$ is in $I(Z)$ if and only if $\text{mult}_{p_i}(f) \geq m_i$ for all $i$, where $\text{mult}_{p_i}(f)$ denotes the multiplicity of $f$ at $p_i$ (this being the least $t$ such that if $l_1$ and $l_2$ are linear forms defining lines which meet at $p_i$ and nowhere else then $f$ is in the $t$th power $(l_1,l_2)^t$ of the ideal ($l_1,l_2) \in R$).

It is important for what follows to note that $I(Z)$ is a homogeneous ideal, hence $I(Z)$ is the direct sum of its homogeneous components $I(Z)_t = R_t \cap I(Z)$ (where for each integer $t$, $R_t$ denotes the $k$-vector space span of all homogeneous polynomials of $R$ of degree $t$).

I.1. Numerical Characters

The work I am interested in here concerns certain numerical characters of ideals $I(m_1p_1 + \cdots + m_np_n) \subset R$ which take a constant value on some nonempty open subset of points $(p_1,\ldots, p_n) \in (P^2)^n$. Thus we will usually consider fat points subschemes $Z = m_1p_1 + \cdots + m_np_n$ for which the points $p_i \in P^2$ are general. (Saying that something is true for $Z = m_1p_1 + \cdots + m_np_n$ for general points $p_i$, is the same as saying that it holds for some open subset of points $(p_1,\ldots, p_n) \in (P^2)^n$.) In order to establish a result for general points, one typically establishes it for some particular special choice of the points and then argues by semicontinuity. (To justify using semicontinuity, even for specializations to infinitely near points, see my 1982 thesis, the relevant parts of which were published in [H2]; alternatively, for specializations keeping the points distinct, see [P].) Thus we will sometimes consider situations for which the points $p_i$ are in some special position.

Given a fat points subscheme $Z = m_1p_1 + \cdots + m_np_n$ and its ideal $I = I(Z)$, among the numerical characters which have been attention by various researchers over the years are the following:

- $\alpha(Z)$, the least degree $t$ such that $I(Z)_t \neq 0$;
- $\beta(Z)$, the least degree $t$ such that the zero locus of $I(Z)_t$ is zero dimensional;
- $h_Z$, the Hilbert function of $I(Z)$ (i.e., the function whose value $h_Z(t)$ for each degree $t$ is the $k$-vector space dimension of $I(Z)_t$);
- $\tau(Z)$, the least degree $t \geq 0$ such that $h_Z(t) = P_Z(t)$, where $P_Z$ is the Hilbert polynomial of $Z$ (which is simply $P_Z(s) = (s^2 + 3s + 2 - \sum_i m_i(m_i + 1))/2$);
- $\nu_t(Z)$, the number of generators of $I(Z)$ in degree $t$ in any minimal set of homogeneous generators;
- $\varepsilon(Z) = \lim_{m\to\infty} \alpha(mZ)/(m(m_1^2 + \cdots + m_n^2))$, the Seshadri constant with respect to $Z$.

(In cases where it is understood which $Z$ is meant, I will sometimes write $\alpha$ or $\beta$, etc., for the more explicit but more cumbersome $\alpha(Z)$, etc.)

The most fundamental characters are $\alpha$, $h$ and $\nu_t$. For example, $h_Z$ immediately determines $\alpha(Z)$ and $\tau(Z)$. Moreover, if one can compute $h_Z$ for any $Z$ then one can also determine $\beta$ for any particular $Z$. (This is because $t < \beta(Z)$ if and only if either $t < \alpha(Z)$, or $t \geq \alpha(Z)$ and there exists some nonzero $Y = m'_1p_1 + \cdots + m'_np_n$ with $0 \leq m'_i \leq m_i$ for all $i$ such that $h_Z(t) = h_{Z-Y}(t - \alpha(Y))$. The idea is that for $\alpha(Z) \leq t < \beta(Z)$, every element of $I(Z)_t$ is divisible by nonconstant homogeneous polynomials $f$ which define divisors in the fixed locus of the linear system $I(Z)_t$. Any such $f$ spans $I(Y)_{\alpha(Y)}$ for an appropriate
Y, as above. Since there are only finitely many \(m'_1p_1 + \cdots + m'_np_n\) with \(0 \leq m'_i \leq m_i\),

one can in principle check whether any such \(Y\) exists, as long as one can always compute \(h\).

Similarly, if one can compute \(\alpha(Z)\) for any \(Z\) then one can also determine \(h_Z\) for any

particular \(Z\). (Here’s how: to compute \(h_Z(t)\) for some \(t\) and some \(Z = m_1p_1 + \cdots + m_np_n\),

let \(Z_0 = Z\) and for each \(j > 0\) let \(Z_j = Z_0 + q_1 + \cdots + q_j\), where \(q_1, \ldots, q_j\) are general.

Since each additional point \(q_i\) imposes one additional condition on forms of degree \(t\) up to

to the point where no forms remain, we see that \(h_Z(t) = i\) where \(i\) is the least \(j\) such that

\(\alpha(Z_j) > t\).

Knowing \(\alpha(Z)\) for a particular \(Z\) sometimes also means we know \(\nu_t(Z)\) for all \(t\).

Indeed, conjectures about the values of \(\alpha\), and, for certain \(Z\), of \(\nu_t\), are made below (see

Conjecture II.3.1). There are examples of \(Z\) for which, if \(\alpha(Z)\) is what it is conjectured to

be, then so are all \(\nu_t(Z)\). (For example, take \(Z = m(p_1 + \cdots + p_n)\) with general points \(p_i\),

where \(n > 9\) is an even square and \(m\) is sufficiently large; see Example 5.2 and Theorem

2.5, both of [HHF].) Nonetheless, knowing \(\alpha\) in general does not seem to be enough to
determine \(\nu_t\) for all \(t\), but if one knows \(\nu_t\) for all \(t\), then one can always compute \(\alpha\) and

hence all of the other characters listed. (This is because the least \(t\) such that \(\nu_t(Z) > 0\)
is \(t = \alpha(Z)\); moreover, \(\nu_{\alpha(Z)}(Z) = h_Z(\alpha(Z))\).) Thus the characters \(\nu_t\) are perhaps even

more fundamental than the other characters discussed above.

The characters \(\nu_t(Z)\) are also interesting due to their connection to minimal free

graded resolutions of \(I(Z)\). A minimal free graded resolution of \(I(Z)\) is an exact sequence

\(0 \rightarrow F_1(Z) \rightarrow F_0 \rightarrow I(Z) \rightarrow 0\) in which \(F_1(Z)\) and \(F_0(Z)\) are free graded \(R\)-modules.

It turns out, up to isomorphism as graded \(R\)-modules, that \(F_0 = \bigoplus_t R[-t]^{\nu_t(Z)}\) and \(F_1(Z)\)
is \(\bigoplus_t R[-t]^{s_t(Z)}\), where the characters \(s_t(Z)\) are defined via \(\nu_t(Z) - s_t(Z) = \Delta^3h_Z(t)\)

[FHH]. Here \(\Delta\) denotes the difference operator (so for any function \(f : Z \rightarrow \mathbb{Z}\),

we have \(\Delta f(t) = f(t) - f(t - 1)\)), and \(R[i]^j\) denotes the direct sum of \(j\) copies of the module \(R\)

itself but taken with the grading defined by \(R[i]^j = R_{t+i}\).

I.2. Connection to Geometry

Additional interest in these characters (and essential techniques in studying them)

comes from their connections to geometry. Given distinct points \(p_1, \ldots, p_n \in \mathbb{P}^2\), let

\(\pi : X \rightarrow \mathbb{P}^2\) be the birational morphism obtained by blowing the points up. Thus \(\pi\) is the

unique morphism where \(X\) is a smooth and irreducible rational surface such that, away

from the points \(p_i\), \(\pi\) is an isomorphism and such that for each \(i\), \(\pi^{-1}(p_i)\) is a smooth

rational curve \(E_i\). It is known that the divisor class group \(\text{Cl}(X)\) is a free abelian group

on the classes \([E_i]\) of the divisors \(E_i\) and on the class \([E_0]\), where \(E_0 = \pi^{-1}(L)\), \(L\) being

any line in \(\mathbb{P}^2\) not passing through any of the points \(p_i\). Thus for any divisor \(D\) on \(X\) we have

\([D] = \sum_{i=0}^n a_i[E_i]\) for some integers \(a_i\).

It will be useful later to recall the intersection form on \(\text{Cl}(X)\). This is a symmetric

bilinear form denoted for elements \([C]\) and \([D]\) of \(\text{Cl}(X)\) by \([C] \cdot [D]\) and determined by

requiring that \([E_i] \cdot [E_j]\) is 0 if \(i \neq j\), 1 if \(i = j = 0\) and \(-1\) if \(i = j > 0\). If \(C\) and \(D\) are

curves on \(X\) such that \(C \cap D\) is finite and transverse, then \([C] \cdot [D] = |C \cap D|\); i.e., \([C] \cdot [D]\)
is just the number of points of intersection of \(C\) with \(D\). If \(C = D\), it is convenient to
denote \([C] \cdot [D]\) by \([C]^2\).

Now, for a divisor \(D\), let \(O_X(D)\) denote the associated line bundle. Given a fat points
subscheme $Z = m_1p_1 + \cdots + m_np_n$, it turns out for all $t$ that $h_Z(t) = h^0(X, \mathcal{O}_X(F_t(Z)))$, where $F_t(Z)$ is the divisor $tE_0 - (m_1E_1 + \cdots + m_mE_n)$ and $h^0(X, \mathcal{O}_X(F_t(Z)))$ denotes the dimension of the 0th cohomology group $H^0(X, \mathcal{O}_X(F_t(Z)))$ of the sheaf $\mathcal{O}_X(F_t(Z))$ (i.e., $h^0(X, \mathcal{O}_X(F_t(Z)))$ is the dimension of the space of global sections of $\mathcal{O}_X(F_t(Z)))$. Thus $\alpha(Z)$ is the least $t$ such that $h^0(X, \mathcal{O}_X(F_t(Z))) > 0$ and $\tau(Z)$ is the least $t \geq 0$ such that $h^0(X, \mathcal{O}_X(F_t(Z))) = P_Z(t)$. Moreover, note that $P_Z(t) = (F_t(Z))^2 - K_X \cdot F_t(Z)/2 + 1$.

By Riemann-Roch we have

$$h^0(X, \mathcal{O}_X(F_t(Z))) - h^1(X, \mathcal{O}_X(F_t(Z))) + h^2(X, \mathcal{O}_X(F_t(Z))) = P_Z(t),$$

and by duality we know $h^2(X, \mathcal{O}_X(F_t(Z))) = 0$ for $t > -3$, so $h^0(X, \mathcal{O}_X(F_t(Z))) - h^1(X, \mathcal{O}_X(F_t(Z))) = P_Z(t)$ for all $t \geq 0$. Thus $h_Z(t) = P_Z(t) + h^1(X, \mathcal{O}_X(F_t(Z))) = \dim R_t - (\sum m_i(m_i + 1)/2 - h^1(X, \mathcal{O}_X(F_t(Z))))$. Now, $I(Z)_t$ is precisely what is left from $R_t$ after imposing for each $i$ the condition of vanishing at $p_i$ to order at least $m_i$; what the previous equation is saying is that the number of conditions imposed is $\sum m_i(m_i + 1)/2 - h^1(X, \mathcal{O}_X(F_t(Z)))$. For all $t$ sufficiently large, $h^1(X, \mathcal{O}_X(F_t(Z))) = 0$ so a total of $\sum m_i(m_i + 1)/2$ conditions are imposed. For smaller $t$, $h^1(X, \mathcal{O}_X(F_t(Z)))$ measures the extent to which these $\sum m_i(m_i + 1)/2$ conditions fail to be independent, and we can regard $\tau(Z)$ as the least degree in which the conditions imposed become independent.

Likewise, the characters $\nu_i$ can be understood from two perspectives. There is a natural map $\mu_i(Z) : I(Z)_t \otimes_k R_1 \to I(Z)_{t+1}$ given by multiplication, and $\nu_{t+1}(Z)$ is just the dimension of the cokernel of the map $\mu_t(Z)$. Corresponding to this map $\mu_t(Z)$ we have in a natural way a map $\mu(F_t(Z)) : H^0(X, \mathcal{O}_X(F_t(Z))) \otimes_k H^0(X, \mathcal{O}_X(E_0)) \to H^0(X, \mathcal{O}_X(F_{t+1}(Z)))$, and indeed $\nu_{t+1}(Z) = \dim \text{cok}(\mu(F_t(Z)))$.

II. Resolutions

Given a fat points subscheme $Z = m_1p_1 + \cdots + m_np_n$, much current work concerns either computing or bounding one or another of the numerical characters cited above. Some of the oldest such work concerned bounding the characters $\nu_t$.

II.1. Dubreil and Campanella Bounds

Dubreil [Dub] obtained two bounds on the minimum number $\sum_i \nu_i(Z)$ of homogeneous generators of $I(Z)$:

**Theorem II.1.1:** Let $Z = m_1p_1 + \cdots + m_np_n$ be a fat points subscheme of $\mathbb{P}^2$ with distinct points $p_i$. Then $\sum_i \nu_i(Z) \leq \alpha(Z) + \beta(Z) - \tau(Z) \leq \alpha(Z) + 1$.

**Sketch of proof:** The inequality $\sum_i \nu_i(Z) \leq \alpha(Z) + 1$ follows immediately from the Hilbert-Burch Theorem. Here is a more elementary proof. Given $R = k[x, y, z]$, we may assume that $x$, $y$ and $z$ define general lines in $\mathbb{P}^2$ (which, in particular, do not contain any of the points $p_i$). It is then easy to see that the image $J = xI(Z)_t + yI(Z)_t$ of the map $xI(Z)_t \oplus yI(Z)_t \to I(Z)_{t+1}$ has dimension $2h_Z(t) - h_Z(t-1)$. Since $J$ has a base point (all elements of $J$ vanish at the common point of vanishing of $x$ and $y$), we see for all $t \geq \alpha(Z)$ that $xI(Z)_t + yI(Z)_t$ cannot contain $zI(Z)_t$. Hence for $t \geq \alpha(Z)$ the image of $\mu_t(Z)$ has
dimension at least $2h_Z(t) - h_Z(t-1) + 1$, hence $\nu_{t+1}(Z) = \text{dim cok}(\mu_t(Z)) \leq h_Z(t+1) - (2h_Z(t) - h_Z(t-1) + 1) = \Delta^2 h_Z(t+1) - 1$, while of course $\nu_1(Z) = h_Z(t) = \Delta^2 h_Z(t)$ for $t = \alpha(Z)$. Summing for $i = \alpha(Z)$ to any $N$ sufficiently large so that $\nu_j(Z) = 0$ and $h_Z(j) = P_2(j)$ for $j > N - 1$, we obtain $\sum_i \nu_i(Z) \leq 1 + \sum_i (\Delta^2 h_Z(t) - 1) = 1 + \Delta h_Z(N) - (N - (\alpha(Z) - 1)) = P_2(N) - P_2(N-1) - N + \alpha(Z) = N + 1 - N + \alpha(Z) = \alpha(Z) + 1$.

The foregoing proof is based on an argument given by Campanella [Cam]. Using a result of [GGR], Campanella there also gives a similar but slightly more refined bound, $\nu_{t+1}(Z) \leq \Delta^2 h_Z(t+1) - \epsilon_t$, where $\epsilon_t$ is 0 for $t < \alpha(Z)$, 1 for $\alpha(Z) \leq t < \beta(Z)$ and 2 for $\beta(Z) \leq t \leq \tau(Z)$. Summing these refined bounds for $i$ from $\alpha(Z)$ to $\tau(Z) + 1$ gives $\sum_i \nu_i(Z) \leq \alpha(Z) + \beta(Z) - \tau(Z)$. (This argument requires that one knows that $\nu_t(Z) = 0$ for $i > \tau(Z) + 1$, but this is true and well known; see [DGM]. It is also not hard to see this directly, at least from the point of view of the surface $X$ obtained by blowing up the points $p_i$. Let $[E_0], \ldots, [E_n]$ be the corresponding basis for $\text{Cl}(X)$, as discussed in Section I. The statement we need to prove is then that $\mu(F_t(Z))$ is surjective if $t > \tau(Z)$. But $t > \tau(Z)$ means $t - 1 \geq \tau(Z)$ and hence $h^1(X, \mathcal{O}_X(F_t-1(Z))) = 0$, so taking global sections of the exact sheaf sequence $0 \to \mathcal{O}_X(F_t-1(Z)) \to \mathcal{O}_X(F_t(Z)) \to \mathcal{O}_{E_0} \otimes \mathcal{O}_X(F_t(Z)) \to 0$, we see $H^0(X, \mathcal{O}_X(F_t(Z)))$ surjects onto $H^0(E_0, \mathcal{O}_{E_0} \otimes \mathcal{O}_X(F_t(Z)))$. But since $E_0$ is isomorphic to $\mathbb{P}^1$, we know that $\mathcal{O}_{E_0} \otimes \mathcal{O}_X(F_t(Z))$ is isomorphic to $\mathcal{O}_{E_0}(t)$ and that $H^0(E_0, \mathcal{O}_{E_0}(1)) \otimes H^0(E_0, \mathcal{O}_{E_0}(t)) \to H^0(E_0, \mathcal{O}_{E_0}(t + 1))$ is surjective, and hence that $H^0(E_0, \mathcal{O}_{E_0}(1)) \otimes H^0(E_0, \mathcal{O}_{E_0} \otimes \mathcal{O}_X(F_t(Z))) \to H^0(E_0, \mathcal{O}_{E_0}(t + 1))$ is surjective. Taking global sections (denoted by $\Gamma$) of $0 \to \mathcal{O}_X \to \mathcal{O}_X(E_0) \to \mathcal{O}_{E_0}(1) \to 0$, tensoring by $V = H^0(X, \mathcal{O}_X(F_t(Z)))$ and mapping by multiplication, one obtains the diagram

\[
\begin{array}{cccc}
0 & \to & \Gamma_X(\mathcal{O}_X) \otimes V & \to & \Gamma_X(\mathcal{O}_X(E_0)) \otimes V & \to & \Gamma_{E_0}(\mathcal{O}_{E_0}(1)) \otimes V & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Gamma_X(\mathcal{O}_X(F_t(Z))) & \to & \Gamma_X(\mathcal{O}_X(F_{t+1}(Z))) & \to & \Gamma_{E_0}(\mathcal{O}_{E_0}(t + 1)) & \to & 0
\end{array}
\]

in which the leftmost vertical map is obviously an isomorphism and the rightmost vertical map as we saw is surjective, so the snake lemma gives an exact sequence in which the cokernels of the outer vertical maps are 0, hence the cokernel cok($\mu(F_t(Z))$) of the middle vertical map also vanishes; i.e., $\nu_t(Z) = 0$ for $t > \tau(Z) + 1$.

In addition to Dubreil’s bounds in Theorem II.1.1, and Campanella’s upper bounds $\nu_{t+1}(Z) \leq \Delta^2 h_Z(t+1) - \epsilon_t$ mentioned in the proof above, Campanella also gave the lower bound that $\nu_t(Z) \geq \max\{\Delta^3 h_Z(t), \epsilon'_t\}$, with $\epsilon'_\beta(Z) = 1$ and $\epsilon'_t = 0$ otherwise (these bounds of course follow from $\nu_t(Z) - s_t(Z) = \Delta^3 h_Z(t)$; Campanella was actually working in the more general situation of perfect codimension 2 subschemes of any projective space).

Here is a result (a slight restatement of Lemma 4.1, [H7]) that in many cases turns out to be an improvement on the bounds above of Dubreil and Campanella, proved in a way very similar to the proof of Theorem II.1.1 given above. It underlies many of the results of [H7], [H8], [HHF] and [FFH].

**Lemma II.1.2:** Let $Z = m_1 p_1 + \cdots + m_n p_n$ be a fat points subscheme of $\mathbb{P}^2$ with distinct points $p_i$ such that $m_1 > 0$, and let $Z'' = Z - p_1$ and $Z' = Z + p_1$. Then $\max\{h_Z(t+1) - 3h_Z(t) + h_{Z''}(t-1), 0\} \leq \nu_{t+1}(Z) \leq h_Z(t+1) - 3h_Z(t) + h_{Z''}(t-1) + h_{Z'}(t)$. 


II.2. Exact Results

Campanella’s and Dubreil’s bounds hold for any \( Z = m_1p_1 + \cdots + m_np_n \subset P^2 \), not just when the points \( p_i \) are general. Thus it is not surprising that the bounds are not always exact. For example, for \( Z = 3(p_1 + \cdots + p_5) \) with \( p_i \) general we have \( \nu_3(Z) = 2 \) but \( \max\{\Delta^3h_Z(8), \epsilon_8^t\} = 1 \) while \( \Delta^2h_Z(8) - \epsilon_7 = 3 \).

Thus we can try to obtain exact results. The typical pattern for work on fat points has been first to obtain results when either the multiplicities \( m_i \) are small or the number \( n \) of points is small, and this is what we see regarding \( \nu \). In particular, for \( Z = p_1 + \cdots + p_n \) with \( p_i \) general, it is easy to see that \( \alpha(Z) \) is the least \( t \) such that \( t^2 + 3t + 2 > 2n \) and that \( \tau(Z) \) is the least \( t \) such that \( t^2 + 3t + 2 \geq 2n \). And, as always, \( \nu_t(Z) = 0 \) unless \( \alpha(Z) \leq t \leq \tau(Z) + 1 \), with, as always, \( \nu_{\alpha(Z)}(Z) = h_Z(\alpha(Z)) \), so for \( Z = p_1 + \cdots + p_n \) only \( \nu_{\alpha(Z)+1}(Z) \) remains to be found, and Geramita, Gregory and Roberts [GGR] proved for such subschemes of general points of \( P^2 \) with multiplicity 1 that \( \mu_{\alpha(Z)} \) has maximal rank (i.e., \( \mu_{\alpha(Z)} \) is either surjective or injective, and hence \( \nu_{\alpha(Z)+1}(Z) = \max\{h_Z(\alpha(Z)+1) - 3h_Z(\alpha(Z)), 0\} \)). Using different methods Idà [Id] has extended this to the case that \( Z = 2(p_1 + \cdots + p_n) \) with \( p_i \) general and \( n > 9 \).

Results have also been obtained for large \( m \) if \( n \) is small. The first such I am aware of is that of Catalisano [Cat2], who determines \( \nu_t(Z) \) for all \( t \) and any \( Z = m_1p_1 + \cdots + m_np_n \) as long as the points \( p_i \) lie on a smooth plane conic; in particular, this handles the case of any \( Z \) involving \( n \leq 5 \) general points.

To discuss Catalisano’s result in more detail, let \( n \) be any positive integer (not necessarily 5 or less) and consider \( Z = m_1p_1 + \cdots + m_np_n \) for any \( n \) distinct points \( p_i \). If \( t \geq \alpha(Z) \), let \( f_t \) be a common factor of greatest degree for the elements of \( I(Z)_t \) (i.e., \( f_t = 0 \) defines the fixed divisor of the linear system of curves given by \( I(Z)_t \)). Now let \( Z'_t = \sum_i (\max\{m_i, \text{mult}_{p_i}(f_t)\})p_i \); thus \( f_t \) spans \( I(Z'_t)_d \), where \( d = \alpha(Z'_t) \) is the degree of \( f_t \). In fact, \( d \) and \( Z'_t \) can be found without finding \( f_t \); \( d = \alpha(Z'_t) \) where among all \( Z'' = m''_1p_1 + \cdots + m''_np_n \) with \( 0 \leq m''_i \leq m_i \) and \( h_Z(t) = h_{Z''}(t - \alpha(Z'')) \) we choose \( Z'_t \) to be that \( Z'' \) for which \( \sum_i (m_i - m''_i) \) is least.

In Catalisano’s situation, the points \( p_i \) are assumed to lie on a smooth conic. By [H1], \( h_Z \) and \( Z'_t \) were already known and easy to compute for points on a conic, and, as noted above, it is enough to determine \( \nu_t(Z) \) for \( t > \alpha(Z) \). Catalisano’s result, although expressed in her paper [Cat2] rather differently, can now be stated:

**Theorem II.2.1:** Let \( p_1, \ldots, p_n \) be distinct points on a smooth plane conic, let \( Z = m_1p_1 + \cdots + m_np_n \) be a fat points subscheme of \( P^2 \) and let \( d = \alpha(Z'_t) \), where \( Z'_t \) is defined as above. Then for each \( t \geq \alpha(Z) \) we have \( \nu_{t+1}(Z) = h_Z(t + 1) - h_{Z - Z'_t}(t + 1 - d) \).

For a proof in a slightly more general case (the conic need not be smooth, for example, and the points can be infinitely near), see [H6]. As an aside, note that Theorem II.2.1 shows that \( \nu_{t+1}(Z) > 0 \) only if \( \alpha(Z) - 1 \leq t < \beta(Z) \), since for \( t \geq \beta(Z) \) we have \( f_t = 1 \), so \( d = 0 \) and \( Z'_t = 0 \).

Since \( n > 5 \) general points do not lie on a conic, Theorem II.2.1 does not apply for \( n > 5 \) general points. Nonetheless, the inequality \( \nu_{t+1}(Z) \geq h_Z(t + 1) - h_{Z - Z'_t}(t + 1 - d) \) always holds (see Lemma 2.10(c), [H6]), although equality can fail for \( n > 5 \) general points since \( \nu_{t+1} \) can be positive even if \( t \geq \beta(Z) \). A complete solution for the case of any
Z = m_1p_1 + ⋯ + m_np_n for n general points p_i was given for n = 6 by Fitchett [F2], for n = 7 by me [H8] and finally for n = 8 by Fitchett, me and Holay [FHH].

Given Z = m_1p_1 + ⋯ + m_np_n with p_i general, the main result of [F2] is that μ(F_t(Z)) has maximal rank as long as F_t(Z) is nef and n ≤ 6. (A divisor D is nef if [D] ⋅ [H] ≥ 0 for every effective divisor H.) Given a divisor on the blow up X of P^2 at n general points, the main results of [H8] give an algorithmic reduction of the problem of determining the rank of μ(F) to the case that F is ample, and shows if n ≤ 7 that μ(F) is surjective as long as F is ample, thereby solving the problem of resolving I(Z) for n ≤ 7. (A divisor D is ample if [D] ⋅ [H] > 0 for every effective divisor H.)

However, if n = 8, reduction to the ample case is not enough, since examples show that μ(F_t(Z)) can fail to have maximal rank even if F_t(Z) is ample; take Z = 4(p_1 + ⋯ + p_7) + p_8 with t = 11, for instance (case (c.ii) of Theorem II.2.2). The main result of [FHH] boils down to giving a formula in nice cases together with an explicit algorithmic reduction to the nice cases. We now give a slightly simplified statement of the main result of [HFF].

Recall that an exceptional curve on a smooth projective surface S is a smooth curve C isomorphic to P^1 such that [C]^2 = −1 in Cl(S). Assuming n = 8, let Ξ_X denote the set of classes of exceptional curves on X and for each exceptional curve C define quantities λ_C and Λ_C as follows: For C = E_i for any i, let λ_C = Λ_C = 0. Otherwise, let m_C be the maximum of C ⋅ E_1, ⋯ , C ⋅ E_n, define Λ_C to be the maximum of m_C and of (C ⋅ L) − m_C and define λ_C to be the minimum of m_C and of (C ⋅ L) − m_C. We then have:

**Theorem II.2.2:** Let X be obtained by blowing up 8 general points of P^2 and let [E_0], [E_1], ⋯ , [E_8] be the associated basis of the divisor class group of X. Consider the class F = t[E_0] − m_1[E_1] − ⋯ − m_8[E_8], where m_1 ≥ ⋯ ≥ m_8.

(a) If F ⋅ C ≥ Λ_C for all C ∈ Ξ_X, then μ(F) has maximal rank.

(b) If F ⋅ C < λ_C for some C ∈ Ξ_X, then ker(μ(F)) and ker(μ(F − C)) have the same dimension.

(c) If neither case (a) nor case (b) obtains, then either

(i) F ⋅ (E_0 − E_1 − E_2) = 0, in which case cok(μ_F) has dimension h^1(X, O_X(F − (E_0 − E_1))) + h^1(X, O_X(F − (E_0 − E_2))), or

(ii) [F] is 3[E_0 − E_1 − ⋯ − E_7] + r[8E_0 − 3E_1 − ⋯ − 3E_7 − E_8] for some r ≥ 1 (in which case dim cok(μ(F)) = r and dim ker(μ(F)) = r + 1), or

(iii) μ(F) has maximal rank.

This theorem leads directly to an algorithm for computing resolutions for fat point subschemes Z involving n ≤ 8 general points of P^2. The MACAULAY 2 script findres included at the end of this paper implements this algorithm to compute the values of ν_t(Z) and h_Z(t) for all α(Z) ≤ t ≤ τ(Z) + 2. Since it does not rely on Gröbner basis computations, it is in comparison quite fast.

**II.3. The Quasi-uniform Resolution Conjecture**

What to expect for n > 8 remains mysterious. Whereas (as discussed below) by taking into account effects due to exceptional curves there results a reasonable conjecture for h_Z for any Z involving general points, doing the same for resolutions is harder. (For a partial result in this direction, see [F1], which at least sharpens the bounds Campanella has given
on $\nu_t(Z)$. Also see Theorem 5.3 of [H7], which shows in a certain sense that behavior in the $n > 8$ case is simple asymptotically, and that it is the case of relatively uniform multiplicities that is not understood.) In fact, the results of [FHH] for $n = 8$ suggest that taking into account the exceptional curves may not be enough. Thus it is still unclear how $\nu_t(Z)$ should be expected to behave in general.

If one puts a mild condition on the coefficients $m_i$, however, there is reason to hope that behavior may be quite simple. In particular, say that $Z = m_1p_1 + \cdots + m_np_n$ is quasi-uniform if: the points $p_i$ are general; $n \geq 9$ and $m_1 = m_9$; and the coefficients $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$ are nonincreasing. We then have the following Quasi-uniform Resolution Conjecture ([HHF]):

**Conjecture II.3.1:** If $Z$ is quasi-uniform, then $h_Z(t) = \max\{P_Z(t), 0\}$ and $\nu_t(Z) = \max\{h_Z(t) - 3h_Z(t - 1), 0\}$ (or, equivalently, $\nu_t(Z) = \max\{0, \Delta^3 h_Z(t)\}$) for all $t \geq 0$.

Assuming this conjecture, we can write down an explicit expression for the resolution of $I(Z)$ for a quasi-uniform $Z$, as follows (see [HHF]):

$$0 \to R[-\alpha - 2]^d \oplus R[-\alpha - 1]^c \to R[-\alpha - 1]^b \oplus R[-\alpha]^a \to I(Z) \to 0,$$

where $\alpha = \alpha(Z)$, $a = h_Z(\alpha)$, $b = \max\{h_Z(\alpha + 1) - 3h_Z(\alpha), 0\}$, $c = \max\{-h_Z(\alpha + 1) + 3h_Z(\alpha), 0\}$, and $d = a + b - c - 1$.

Most of the evidence for this conjecture currently is for the uniform case (i.e., $Z$ is quasi-uniform with all multiplicities $m_i$ equal to some single $m$). See Figure 1 for a graph showing all $m$ and $n$ up to about 450 such that the resolution of $I(m(p_1 + \cdots + p_n))$ is known and by whom; in all cases with $n \geq 9$, the resolution is that given by Conjecture II.3.1. (Figure 1 is a color postscript graphic but your copy may have printed out in grayscale; a color version can be viewed at [http://www.math.unl.edu/~bharbour/ResAll.jpg](http://www.math.unl.edu/~bharbour/ResAll.jpg). For example, the conjecture is true whenever $m \leq 2$ (see [GGG] and [Id]), and easy for $n = 9$ (see [H6]).

In addition, there are two situations in which the conjecture is known for unbounded multiplicities. For example, it is always true that $\nu_t(Z) = 0$ unless $\alpha(Z) = t \leq \tau(Z) + 1$, so whenever it can be shown that $\alpha(Z) > \tau(Z)$ (such as is the case in characteristic 0 when $Z = m(p_1 + \cdots + p_n)$ for $n = d^2 + 2k$ general points $p_i$ with $m = d(d \pm 1)/(2k)$, for any integers $d > 2$ and $k$ such that $m$ is an integer; see [HR], which applies the modified unloading method due to Roé and me, discussed below in Section IV.1.3 and at the end of Section IV.2), it follows that $I(Z)$ is generated in degree $\alpha(Z)$ and the resolution is as in the preceding paragraph; explicitly,

$$0 \to R[-\alpha - 1]^\alpha \to R[-\alpha]^\alpha + 1 \to I(Z) \to 0.$$

The only other cases known (see [HR], and Corollary 1.1 of [HHF]) are obtained by applying the results of [HHF]. For example, if $n$ is an even square and $m \geq (\sqrt{n} - 2)/4$, [HHF] shows that Conjecture II.3.1 is true for $Z = m(p_1 + \cdots + p_n)$ whenever $\alpha(Z) = \tau(Z)$. But (in characteristic 0) [Ev2] shows that $\alpha(Z) = \tau(Z)$ whenever $n$ is a square divisible by no primes greater than 5 (although full proof is given only for the case that $n$ is a power of 4), and [HR] shows (in characteristic 0) that $\alpha(Z) = \tau(Z)$ for infinitely many $m$ for any even square $n$. Additional cases in which the resolution is determined by applying the results
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Figure 1a: Graph showing for which $(n,m)$ the resolution of the ideal $I(n,m)$ of $n$ general points in $\mathbb{P}^2$ of multiplicity $m$ is known, and by whom, as of 1999.

Figure 1b: Graph showing for which $(n,m)$ the resolution of the ideal $I(n,m)$ of $n$ general points in $\mathbb{P}^2$ of multiplicity $m$ is now known via Harbourne-Roé.
of [HHF] but for which \( n \) is not a square are given by [HR]. Further evidence of various kinds for the conjecture is also given in [HHF].

In order to facilitate checking this conjecture and exploring the problem of understanding resolutions when \( Z \) need not be quasi-uniform, it is helpful to be able to compute resolutions directly. Although we are interested in general points, it is easiest instead to use random choices of points, with the expectation that this will usually give points that are general enough. It is possible to implement such a calculation very simply in MACAULAY. Here is an example of such a MACAULAY 2 script, provided to me by Hal Schenck, for computing resolutions of ideals \( I(\sum m_i p_i) \) for random choices of points \( p_i \in \mathbb{P}^2 \):

\[
R = \mathbb{Z}/31991[x_0..x_2]
\]

\[
mixer = (1)->(i:=0;
\quad b:=\text{ideal (matrix \{\{1\}\}*R)};
\quad \text{scan}#1, i->(
\quad \quad f:=\text{random}(R^1,R^{-1});
\quad \quad g:=\text{random}(R^1,R^{-1});
\quad \quad I:=(\text{ideal (f | g)})^{\text{(1#i)}};
\quad \quad b=\text{intersect}(b,I));
\quad \text{print betti res coker gens b;}
\quad b)
\]

\(--\text{Return the ideal of mixed multiplicity random fatpoints. Input is a list}\
\quad --\text{with the multiplicities; e.g. mixer(\{1,2,3\}) returns the ideal of}\
\quad --I(p_1)^1 \cap I(p_2)^2 \cap I(p_3)^3, where p_j is a (random) point in P^2,}\
\quad --\text{and prints the betti numbers of the resolution}\
\quad --\text{HKS 4/28}\)

By a slight modification, below, this script can be made to handle the uniform case (i.e., \( n \) random points each taken with the same multiplicity \( m \)):

\[
R = \mathbb{Z}/31991[x_0..x_2]
\]

\[
unif = (n,m)->(<< n << " points of multiplicity " << m << ":" << endl;
\quad i:=0;
\quad b:=\text{ideal (matrix \{\{1\}\}*R)};
\quad \text{while } i < n \text{ do (}
\quad \quad f:=\text{random}(R^1,R^{-1});
\quad \quad g:=\text{random}(R^1,R^{-1});
\quad \quad I:=(\text{ideal (f | g)})^{(m)};
\quad \quad b=\text{intersect}(b,I);
\quad \quad i=i+1);
\quad \text{print betti res coker gens b;}
\quad b)
\]

\(--\text{Example: unif(\{3,2\}) returns the ideal of}\
\]
--I(p1)\cap I(p2)\cap I(p3), where pj is a (random) point in P^2, --and prints the betti numbers of the resolution

### III. Hilbert functions

Nagata, in connection with his work on Hilbert’s 14th problem, began an investigation of the Hilbert function $h_Z$ for fat points subschemes $Z = m_1p_1 + \cdots + m_np_n$ with $p_i$ general, although his work was written up from the point of view of divisors on blow ups of $\mathbb{P}^2$ (see [N2]).

#### III.1. Nagata’s Work

In brief, Nagata in [N2] determines $h_Z(t)$ (and thus $\alpha(Z)$) for all $t$ for any $Z = m_1p_1 + \cdots + m_np_n$ with $p_i$ general as long as $n \leq 9$. In [N1], he poses the following conjecture, which remains open unless $n$ is a square, in which case Nagata verified it:

**Conjecture III.1.1:** Let $Z = m_1p_1 + \cdots + m_np_n$ for $n > 9$ general points $p_i \in \mathbb{P}^2$. Then $\alpha(Z) > (m_1 + \cdots + m_n)/\sqrt{n}$.

As Nagata points out, it is enough to consider the uniform case. Keeping in mind that Nagata proved $\alpha(Z) > m\sqrt{n}$ in the case that $n$ is a square, Conjecture III.1.1 is thus equivalent to:

**Conjecture III.1.2:** Let $Z = m(p_1 + \cdots + p_n)$ for $n > 9$ general points $p_i \in \mathbb{P}^2$. Then $\alpha(Z) \geq m\sqrt{n}$.

Restated in terms of $n > 9$ generic points, this is equivalent to:

**Conjecture III.1.3:** Let $Z = p_1 + \cdots + p_n$ for $n > 9$ generic points $p_i \in \mathbb{P}^2$. Then $\varepsilon(Z) = 1/\sqrt{n}$.

Also implicit in [N2] is a lower bound (see (*), Section III.2) for the values of the Hilbert function $h_Z$ of $I(Z)$. An easier lower bound comes from the fact, as discussed above, that $h_Z(t) = F_Z(t) + h^1(X, \mathcal{O}_X(F_t(Z)))$. Since $h^1(X, \mathcal{O}_X(F_t(Z))) \geq 0$, it of course follows that $h_Z(t) \geq \max\{P_Z(t), 0\}$. However, easy examples show that $h_Z(t) - \max\{P_Z(t), 0\}$ can sometimes occur; in all such examples for which $h_Z(t)$ is known, the difference $h_Z(t) - \max\{P_Z(t), 0\}$ has a geometric origin, being always precisely what one gets by taking into account exceptional curves. Taking the exceptional curves into account gives the more refined bound (*).

To explain this, let $X$ be obtained by blowing up $n$ distinct points $p_i$ of $\mathbb{P}^2$. We have, as discussed above, the basis $[E_0], \ldots, [E_n]$ of the divisor class group $\text{Cl}(X)$ of $X$. Because we are mostly interested in the case of $n$ general points, technical issues force us to use the following definition. Let us say that an element $v = \sum_i a_i[E_i]$ of $\text{Cl}(X)$ is an *exceptional class* if for general points $p_i$ there is an exceptional curve $C \subset X$ with $v = [C]$. (The problem is that there may be no nonempty open set $U$ of points $(p_1, \ldots, p_n) \in (\mathbb{P}^2)^n$ for which all exceptional classes are simultaneously classes of exceptional curves, even though each exceptional class $v$ is the class of an exceptional curve for some nonempty open $U_v \subset (\mathbb{P}^2)^n$.)
Nagata [N2] determined the set \( \mathcal{E}(n) \) of exceptional classes. It turns out that \( \mathcal{E}(0) \) is empty, \( \mathcal{E}(1) = \{[E_1]\} \), and \( \mathcal{E}(2) = \{[E_1], [E_2], [E_0 - E_1 - E_2]\} \), while for \( n \geq 3 \) the set \( \mathcal{E}(n) \) is the orbit \( W_n[E_n] \) with respect to the action of the group \( W_n \) of linear transformations on \( \text{Cl}(X) \) generated by all permutations of \( \{[E_1], \ldots, [E_n]\} \) and (if \( n \geq 3 \)) by the map \( \gamma \) for which \( \gamma : [E_i] \mapsto [E_i] \) for \( i > 3 \), \( \gamma : [E_i] \mapsto [E_0] - [E_1] - [E_2] - [E_3] + [E_i] \) for \( 0 < i \leq 3 \) and \( \gamma : [E_0] \mapsto 2[E_0] - [E_1] - [E_2] - [E_3] \). (The map \( \gamma \) can be regarded as a reflection corresponding in an appropriate sense to a quadratic Cremona transformation centered at \( p_1, p_2 \), and \( p_3 \). The fact that \( W_n \) is a reflection group was recognized by Du Val [DuV2], and extended and exploited by Looijenga [L].)

### III.2. A Decomposition and Lower Bound

For simplicity, assume \( n \geq 3 \). This will not be a serious restriction, since cases \( n < 3 \) are easy to handle ad hoc, and in any case there are natural inclusions \( \mathcal{E}(n) \subset \mathcal{E}(n+1) \) for all \( n \), so a given value of \( n \) subsumes smaller values. Now let \( \Psi \) be the subsemigroup of \( \text{Cl}(X) \) generated by \( \mathcal{E}(n) \) and by the anticanonical class \( -K_X = 3E_0 - E_1 - \cdots - E_n \) of \( X \). (With respect to the action of \( W_n \) on \( \text{Cl}(X) \), \( \Psi \) is essentially Tits’ cone [Ka]; thus there exists a fundamental domain for the action of \( W_n \) on \( \Psi \).) For any \( F \in \Psi \), it turns out that there is a unique decomposition \( F = H_F + N_F \) with (dropping the subscripts) \( H, N \in \Psi \) such that \( H \cdot v \geq 0 \) for every exceptional class \( v \), \( H \cdot N = 0 \), and either \( N = 0 \) or \( N = a_1 v_1 + \cdots + a_r v_r \) for some exceptional classes \( v_i \) and integers \( a_i \geq 0 \), such that \( v_i \cdot v_j = 0 \) for all \( i \neq j \). (It is easy to compute this decomposition. By recursively applying \( \gamma \) and permutations in a straightforward way, for any \( F \in \text{Cl}(X) \) one can find an element \( w \in W_n \) such that either \( wF \cdot [E_0] < 0 \), or \( wF \cdot [E_0 - E_1] < 0 \), or such that \( wF = a_0 [E_0] + \sum_{i>0} a_i [E_i] \) with \( a_0 \geq 0 \), \( a_0 + a_1 + a_2 + a_3 \geq 0 \) and \( a_1 \leq \cdots \leq a_n \). But if either \( wF \cdot [E_0] < 0 \) or \( wF \cdot [E_0 - E_1] < 0 \), then \( F \notin \Psi \), while otherwise there are two cases. Either \( 0 > wF \cdot [E_0 - E_1 - E_2] = a_0 + a_1 + a_2 \), in which case \( H = w^{-1}((2a_0 + a_1 + a_2)E_0 - (a_0 + a_2)E_1 - (a_0 + a_1)E_2) \) and \( N = w^{-1}((-a_1 - a_2 - a_0)[E_0 - E_1 - E_2] + \sum_{i>0} a_i [E_i]) \), or \( wF \cdot [E_0 - E_1 - E_2] \geq 0 \) and we have \( H = w^{-1}(a_0 [E_0] + \sum_{i>0} a_i [E_i]) \) and \( N = w^{-1}(\sum_{i>0} a_i [E_i]) \). An implementation of this procedure is given by the script \texttt{decomp} provided in this paper.)

It is true (and more or less apparent from [N2]) for general points \( p_i \) that if \( h^0(X, \mathcal{O}_X(F)) > 0 \) then \( F \in \Psi \), hence \( F = H + N \) as above, and \( h^0(X, \mathcal{O}_X(F)) = h^0(X, \mathcal{O}_X(H)) \geq (H^2 - K \cdot H)/2 + 1 \). For any \( F \in \text{Cl}(X) \), define \( e(F) \) to be 0 unless \( F \in \Psi \), in which case set \( e(F) \) to be the maximum of \( 1 + (H_F^2 - K \cdot H_F)/2 \) and 0. We then get the lower bound

\[
h^0(X, \mathcal{O}_X(F)) \geq e(F). \tag{*}
\]

(The script \texttt{homcompdim} included at the end of this paper computes \( e(F) \). For \( F = d[E_0] - m_1[E_1] - \cdots - m_n[E_n] \), we have \( e(F) = \text{homcompdim}(\{d, \{m_1, \ldots, m_n\}\}) \), which, if the multiplicities \( m_i \) are nonnegative, is also thus a lower bound for the dimension of the homogeneous component of \( I(Z) \) of degree \( d \) for \( Z = m_1 p_1 + \cdots + m_n p_n \).)

### III.3. The SHGH Conjectures

It follows from Nagata’s work (see Theorem 9, [N2]) that in fact \( h^0(X, \mathcal{O}_X(F)) = e(F) \) for \( n \leq 9 \) general points. What occurs for \( n > 9 \) is not known, but I [H1] (also see [H3]),
Gimigliano [Gi1] (also see [Gi2]) and Hirschowitz [Hi2] independently gave conjectures for explicitly computing $h^0(X, \mathcal{O}_X(F))$ for any $n$. These conjectures are all equivalent to the following conjecture, which states that $e(F)$ is the expected value of $h^0(X, \mathcal{O}_X(F))$:

**Conjecture III.3.1:** Let $X$ be the blow up of $n$ general points of $\mathbb{P}^2$ and let $F \in \text{Cl}(X)$. Then $h^0(X, \mathcal{O}_X(F)) = e(F)$.

It is interesting to compare Conjecture III.3.1 with an earlier conjecture posed by Segre [Seg], giving a conjectural characterization of those classes $F$ such that $h^0(X, \mathcal{O}_X(F)) > \max\{0, (F^2 - K \cdot F)/2 + 1\}$:

**Conjecture III.3.2:** Let $X$ be the blow up of $n$ general points of $\mathbb{P}^2$ and let $F \in \text{Cl}(X)$. If $h^0(X, \mathcal{O}_X(F)) > \max\{0, (F^2 - K \cdot F)/2 + 1\}$, then the fixed locus of $|F|$ has a double component.

It is easy to show that Conjecture III.3.1 implies Conjecture III.3.2; the fact that Conjecture III.3.2 implies Conjecture III.3.1 is essentially Theorem 8 of [N2]. Thus I will refer to these conjectures (in any of their forms) as the SHGH Conjecture. Since Nagata’s paper is hard to read, the equivalence of Conjecture III.3.1 and Conjecture III.3.2 was only recently recognized and proved by Ciliberto and Miranda. Here is a sketch of a proof.

**Theorem III.3.3:** Conjecture III.3.1 is equivalent to Conjecture III.3.2.

**Sketch of proof:** To see that Conjecture III.3.1 implies Conjecture III.3.2, assume that $h^0(X, \mathcal{O}_X(F)) > \max\{0, (F^2 - K \cdot F)/2 + 1\}$ for some $F$. Thus $h^0(X, \mathcal{O}_X(F)) > 0$, so $F \in \Psi$ and hence we have a decomposition $F = H + N$, as described above, with $N = a_1v_1 + \cdots + a_nv_n$ for some exceptional classes $v_i$ and $a_i \geq 0$. By Conjecture III.3.1, we have $0 < h^0(X, \mathcal{O}_X(F)) = e(F) = (H^2 - K \cdot H)/2 + 1$, and by substituting $H + N$ in for $F$ we see that $(F^2 - K \cdot F)/2 + 1 = (H^2 - K \cdot H)/2 + 1$ unless $a_i > 1$ for some $i$, in which case $v_i$ is the class of a curve occurring (at least) doubly in the base locus of $|F|$, proving Conjecture III.3.2.

Conversely, assume Conjecture III.3.2. Among all $F$ for which $h^0(X, \mathcal{O}_X(F)) = e(F)$ fails, choose one having as few fixed components as possible (i.e., for which the sum of the multiplicities of the fixed components is minimal). As before we have $F = H + N$, but $N = 0$ by minimality (since $h^0(X, \mathcal{O}_X(H)) = h^0(X, \mathcal{O}_X(F)) > e(F) = e(H)$). Since $F = H$, by construction of $H$ we have $F \cdot E \geq 0$ for every exceptional class $E$.

Now say some reduced irreducible curve $C$ occurs as a fixed component of $|F|$ with multiplicity at least 2. Thus $h^0(X, \mathcal{O}_X(2C)) = 1$, hence $h^0(X, \mathcal{O}_X(C)) = 1$ and by Conjecture III.3.2 we have $(C^2 - C \cdot K)/2 + 1 = 1$ so $C^2 = C \cdot K$. Therefore the genus $g_C$ of $C$ is $(C^2 + C \cdot K)/2 + 1 = C^2 + 1$; i.e., $C^2 \geq -1$. On the other hand $1 = h^0(X, \mathcal{O}_X(2C)) \geq (4C^2 - 2C \cdot K)/2 + 1 = C^2 + 1$ so $C^2 \leq 0$.

If $C^2 = -1$, then $g_C = 0$, so $C$ is an exceptional curve. From $0 \rightarrow \mathcal{O}_X(F - C) \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_C(C \cdot F) \rightarrow 0$ it follows that $h^1(X, \mathcal{O}_X(F)) = 0$ (since $C \cdot F \geq 0$ implies $h^1(X, \mathcal{O}_C(C \cdot F)) = 0$, while $h^1(X, \mathcal{O}_X(F - C)) = 0$ by minimality), which contradicts failure of $h^0(X, \mathcal{O}_X(F)) = e(F)$.

If $C^2 = 0$, then $g_C = 1$, so $C$ is an elliptic curve. From $0 \rightarrow \mathcal{O}_X(F - C) \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_C \otimes \mathcal{O}_X(F) \rightarrow 0$ it follows that $h^1(X, \mathcal{O}_X(F)) = 0$ (as long as we see $h^1(X, \mathcal{O}_C \otimes \mathcal{O}_X(F)) = 0$, since $h^1(X, \mathcal{O}_X(F - C)) = 0$ by minimality), which again contradicts failure
of $h^0(X, \mathcal{O}_X(F)) = e(F)$. But $C$ (being irreducible of nonnegative selfintersection) is obviously nef, so $C \cdot F \geq 0$. Since $C$ is elliptic, $C \cdot F \geq 0$ guarantees $h^1(X, \mathcal{O}_C \otimes \mathcal{O}_X(F)) = 0$ unless the restriction $\mathcal{O}_C \otimes \mathcal{O}_X(F)$ of $\mathcal{O}_X(F)$ to $C$ is trivial. But because the points blown up to obtain $X$ are general, $\mathcal{O}_C \otimes \mathcal{O}_X(F)$ cannot be trivial. (In fact, up to Cremona transformations, which is to say up to the action of $W_n$, we can assume that $[C] = [3E_0] − [E_1 + \cdots + E_9]$. For $\mathcal{O}_C \otimes \mathcal{O}_X(F)$ to be trivial we would need $F \cdot C = 0$ and, writing $[F]$ as $[dE_0 − m_1E_1 − \cdots − m_nE_n]$, we would have $3d − (m_1 + \cdots + m_9) = 0$; i.e., $[dE_0 − m_1E_1 − \cdots − m_9E_9]$ is the class of an effective divisor perpendicular to $[3E_0] − [E_1 + \cdots + E_9]$, but for general points $p_1, \ldots, p_9$ the only such classes are multiples of $[3E_0] − [E_1 + \cdots + E_9]$ itself, for which it is easy to check the restrictions to $C$ are not, in general, trivial.)

III.4. Evidence

It is worth mentioning that it is not hard to show (see [HHF]) that the SHGH Conjecture implies that $h_Z(t) = \max\{P_Z(t), 0\}$ if $Z$ is quasi-uniform, which is part of Conjecture II.3.1 posed above. In particular, if $F = d[E_0] − m[E_1 + \cdots + E_n]$ where the $E_i$ are obtained by blowing up $n > 9$ general points of $\mathbb{P}^2$, then Conjecture III.3.1 predicts that $h^0(X, \mathcal{O}_X(F))$ equals the maximum of 0 and $(F^2 − K \cdot F)/2 + 1$. Proving this equality is trivial if $m = 1$, and was proved for $m \leq 3$ by Hirschowitz [Hi1]. (It is worth noting that in a series of papers culminating in [AH1], Alexander and Hirschowitz handle the case of $m = 2$ for $\mathbb{P}^N$ for all $N$.) More generally, given any positive integer $M$ and $M \geq m_i > 0$ for all $i$, [AH2] shows for any $Z = m_1p_1 + \cdots + m_np_n$ (in any projective space) that $h_Z(t) = \max\{P_Z(t), 0\}$ for all $t$ as long as $n$ is sufficiently large compared with $M$.

More explicitly, Ciliberto and Miranda [CM1], [CM2] have verified the SHGH Conjecture in characteristic 0 for all $m \leq 12$ for any $n > 9$ (see also [Sei]), and Mignon [Mi] has now verified the SHGH Conjecture for all $n > 9$ for any $F = d[E_0] − m_1E_1 − \cdots − m_nE_n$ as long as $m_i \leq 4$ for all $i$.

Whereas all of the explicit verifications of the SHGH conjecture described above assume multiplicities at most 12, two methods now exist that work for multiplicities which in some cases can be arbitrarily large; both assume that the characteristic is 0. The first is the recent result of Evain [Ev2], which, for example, shows that $h^0(X, \mathcal{O}_X(F))$ equals the maximum of 0 and $(F^2 − K \cdot F)/2 + 1$ for any $F = d[E_0] − m[E_1 + \cdots + E_n]$ as long as $X$ is obtained by blowing up $n$ general points with $n$ being a power of 4. The second is the modified unloading method [HR] jointly due to me and J. Roë (see Section IV.1.3 and the end of Section IV.2), which gives very tight bounds on $\alpha$ and $\tau$. With good enough bounds, one can sometimes show $\alpha(Z) \geq \tau(Z)$, but anytime one knows $\alpha(Z) \geq \tau(Z)$ it immediately follows that the SHGH Conjecture holds for $Z$. In fact, there are numerous examples for which the bounds from [HR] are good enough to show $\alpha(Z) \geq \tau(Z)$ and hence that the SHGH Conjecture holds, including certain infinite families of examples $Z = m(p_1 + \cdots + p_n)$ such as for infinitely many $m$ when $n$ is any square (see [HR]), or $n = d(d + 1)$, $m = d + 1$ for any even integer $d > 2$ (mentioned in Section II.3) or $n = d^2 + 2$ and $m = d(d^2 + 1) + d(d + 1)/2$ for any $d > 2$ (see [HR] for these and other examples). See Figure 2 for a graph showing all $m$ and $n$ up to about 400 such that the Hilbert function of $I(m(p_1 + \cdots + p_n))$ is known and by whom; in all cases the Hilbert function is that given by the SHGH conjecture. (Figure 2 is a color postscript
graphic but your copy may have printed out in grayscale; a color version can be viewed at
http://www.math.unl.edu/~bharbour/HilbAll2.jpg.

We close this section with the comment that the script findhilb computes the SHGH
conjectural values of the Hilbert function of \( I(m_1p_1 + \cdots + m_np_n) \) for general points \( p_i \).
For \( n < 10 \), these values are the actual values. No separate script for the case of uniform
multiplicities is included since for \( Z = m(p_1 + \cdots + p_n) \) for \( n > 9 \) general points \( p_i \),
the conjecture is simply that \( h_Z(t) \) is the maximum of 0 and \( (t^2 + 3t + 2 - nm(m + 1))/2 \).

IV. Bounds

Rather than trying to prove the SHGH Conjecture directly, a good deal of work has
been directed toward obtaining better bounds on \( \alpha \) and \( \tau \). The values of \( \alpha \) and \( \tau \) predicted
by the SHGH Conjecture give upper and lower bounds, respectively; in particular, \( \alpha(Z) \)
is less than or equal to the least \( t \) such that \( e(F_t(Z)) > 0 \), and \( \tau(Z) \) is greater than or
equal to the least \( t \geq 0 \) such that \( e(F_t(Z)) = P_2(t) \). Thus what is of most interest are
lower bounds on \( \alpha \) and upper bounds on \( \tau \). Bounds on \( \alpha \) are especially of interest, since
a sufficiently good lower bound on \( \alpha \) may equal the upper bound (and presumed actual
value) of \( \alpha \) given by the SHGH Conjecture, and, as discussed above, if \( \alpha \) always has its
conjectured value then the full SHGH Conjecture is true.

Unfortunately, such tight bounds are so far fairly rare, but there are some cases, as
discussed above, for which (by precisely this method of tight bounds) the Hilbert function
and resolution are known. For two additional examples, consider \( F = d[E_0] - m[E_1 +
\cdots + E_n] \) where \( X \) is obtained by blowing up \( n \) general points of \( \mathbb{P}^2 \) with \( n \) being either
16 or 25. Although Evain’s method handles these cases, at least in characteristic 0, an
alternate approach is to notice that the inequality \( h^0(X, \mathcal{O}_X(F)) \geq (F^2 - K \cdot F)/2 + 1 \)
with \( F = d[E_0] - m[E_1 + \cdots + E_n] \) guarantees that \( \alpha \leq \lfloor m\sqrt{n} + (\sqrt{n} - 3)/2 \rfloor \).
For \( n = 16 \) general points this gives \( \alpha \leq 4m + 1 \) while for \( n = 25 \) this gives \( \alpha \leq 5m + 1 \). But Nagata’s
result [N1] that \( \alpha > m\sqrt{n} \) when \( n \) is a square bigger than 9 now shows that \( \alpha = 4m + 1 \)
for \( n = 16 \) and \( \alpha = 5m + 1 \) for \( n = 25 \). By [HFF], \( \tau = \alpha \) in these cases, which for all \( m \)
and \( t \) determines \( h_Z \) (and even the resolution of \( I(Z) \) when \( n = 16 \)).

Some of the bounds discussed below are algorithmic in nature, and hard to give simple
explicit formulas or estimates for. Thus, to compute them, I have included at the end of
this paper two MACAULAY 2 scripts, bounds(l) and unifbounds(l); in the former case
\( l = \{m_1,\ldots,m_n\} \) (corresponding to taking \( n \) general points with multiplicities \( m_1,\ldots,m_n \))
while in the latter case \( l = \{n,m\} \) (corresponding to taking \( n \) general points each with
multiplicity \( m \)).

IV.1. Bounds on \( \alpha \)

By Nagata’s work [N2], the exact value of \( \alpha(Z) \) is known for any \( Z = m_1p_1 + \cdots + m_np_n \)
with \( p_i \) general and \( n \leq 9 \), and in such cases can be computed by running the script
findalpha or uniffindalpha. For \( Z = m(p_1 + \cdots + p_n) \), \( n \leq 9 \), it is easy to be explicit:
\( \alpha(Z) = [c_n m] \), where \( c_1 = c_2 = 1 \), \( c_3 = 3/2 \), \( c_4 = c_5 = 2 \), \( c_6 = 12/5 \), \( c_7 = 21/8 \), \( c_8 = 48/17 \)
and \( c_9 = 3 \).

For \( n > 9 \), findalpha or uniffindalpha only give upper bounds for \( \alpha \), although
the upper bounds given should be, according to the SHGH Conjecture, the actual values.
Figure 2a: Graph showing for which \((n,m)\) the Hilbert function of the ideal \(I(n,m)\) of \(n\) general points in \(\mathbb{P}^2\) of multiplicity \(m\) is known, and by whom, as of 1999.

- \(m \leq 12\): Ciliberto-Miranda: 1998
- \(m \leq 12\): Hirschowitz: 1985
- \(n \leq 9\): Nagata: 1960
- \(n=4\): Evain: 1999

Figure 2b: Graph showing for which \((n,m)\) the Hilbert function of the ideal \(I(n,m)\) of \(n\) general points in \(\mathbb{P}^2\) of multiplicity \(m\) is known via Harbourne-Roé.
Thus most interest is in finding lower bounds on \( \alpha \), and a number of such have been given. Let \( n \geq n' > 9 \) and consider \( Z = m(p_1 + \cdots + p_n) \) and \( Z' = m(p_1 + \cdots + p_{n'}) \), where the points \( p_i \) are general. It is easy to see that \( \alpha(Z) \geq \alpha(Z') \). Since Nagata [N1] proves that \( \alpha(Z') > m\sqrt{n'} \) if \( n' \) is a square, it follows (taking \( n' = \lceil \sqrt{n} \rceil^2 \) when \( n \) is 16 or more) that \( \alpha(Z) > m\lceil \sqrt{n} \rceil \). A complete proof is somewhat tricky; we treat the slightly weaker inequality \( \alpha(Z) \geq m\lceil \sqrt{n} \rceil \) in the next section.

IV.1.1. Bounds by testing against nef divisors

The inequality \( \alpha(Z) \geq m\lceil \sqrt{n} \rceil \) follows easily (for any \( n \)) by specializing \( \lceil \sqrt{n} \rceil^2 \) of the points \( p_i \) to a smooth plane curve \( C' \) of degree \( \lfloor \sqrt{n} \rfloor \). The class \( C \) of the proper transform of \( C' \) to the blow up \( X \) of \( \mathbb{P}^2 \) at the points \( p_i \) is \( \lfloor \sqrt{n} \rfloor [E_0] - ([E_1 + \cdots + E_{n'}]) \), which is nef, but \( \alpha(Z)[E_0] - m[E_1 + \cdots + E_n] \) is (by definition of \( \alpha(Z) \)) the class of an effective divisor, so the intersection \( C : (\alpha(Z)[E_0] - m[E_1 + \cdots + E_n]) = \alpha(Z)[\lfloor \sqrt{n} \rfloor] - m\lceil \sqrt{n} \rceil^2 \) is nonnegative, which gives \( \alpha(Z) \geq m\lceil \sqrt{n} \rceil \). More generally, the same argument works for \( Z = m_1 p_1 + \cdots + m_n p_n \), giving \( \alpha(Z) \geq (m_1 + \cdots + m_{\lceil \sqrt{n} \rceil})/\lceil \sqrt{n} \rceil \).

Alternatively, by specializing all \( n \) points to a curve of degree \( \lceil \sqrt{n} \rceil \), the same argument (using the fact that now \( \lfloor \sqrt{n} \rfloor [E_0] - [E_1 + \cdots + E_n] \) is nef) gives the inequality \( \alpha(Z) \geq mn/\lfloor \sqrt{n} \rfloor \) for \( Z = m(p_1 + \cdots + p_n) \), and \( \alpha(Z) \geq (m_1 + \cdots + m_n)/\lceil \sqrt{n} \rceil \) for \( Z = m_1 p_1 + \cdots + m_n p_n \). More generally, we have the following extension of the main result of [H9] (for a further generalization, see [H10]):

**Theorem IV.1.1.1:** Let \( Z = m_1 p_1 + \cdots + m_n p_n \) for general points \( p_i \in \mathbb{P}^2 \) with \( n \geq 1 \) and \( m_1 \geq \cdots \geq m_n \), and let \( r \leq n \) and \( d \) be positive integers. Given nonnegative rational numbers (not all 0) \( a_0 \geq a_1 \geq \cdots \geq a_n \geq 0 \) such that \( a_0 d^2 \geq a_1 + \cdots + a_r \) and \( ra_0 \geq a_1 + \cdots + a_n \), then \( \alpha(Z) \geq (\sum_i a_i m_i)/(a_0 d) \).

**Sketch of proof:** Note that by multiplying by a common denominator, we may assume that each \( a_i \) is a nonnegative integer. Consider the class \( F = [a_0 d E_0 - a_1 E_1 - \cdots - a_n E_n] \) on the surface \( X \) obtained by blowing up the points \( p_i \). First, specialize (as in the proof of the main result of [H9]) to certain infinitely near points; in particular, such that \( [E_i - E_{i+1}] \) for each \( 0 < i < n \) is the class of an effective, irreducible divisor on the specialization \( X' \) of \( X \), and such that \( d[E_0] - [E_1 + \cdots + E_r] \) is the class \( C \) of the proper transform of a smooth plane curve. Now \( F \) is nef on \( X' \) and hence on \( X \). To see this, note that: \( F \cdot C \geq 0 \) since \( a_0 d^2 \geq a_1 + \cdots + a_r \); \( F \cdot (E_i - E_{i+1}) \geq 0 \) for all \( i > 0 \) since \( a_i \geq a_{i+1} \); \( F \cdot E_n \geq 0 \) since \( a_n \geq 0 \); and \( F \) is a nonnegative integer sum of the classes \( C, [E_i - E_{i+1}], i > 0, \) and \( [E_n] \) since \( a_0 \geq a_1 \) and \( ra_0 \geq a_1 + \cdots + a_n \). Thus \( F \) is a sum of effective classes (in particular, of \( a_0 C \) and various multiples of the \( [E_i - E_{i+1}] \) and \( E_n \)), each of which it meets nonnegatively; thus \( F \) is nef and so \( F \) meets \( \alpha(Z)E_0 - m_1 E_1 - \cdots - m_n E_n \) nonnegatively, from which our result follows.

Finding an optimal bound for a given \( Z \) using Theorem IV.1.1.1 involves solving a linear programming problem (note that we may normalize so that \( a_0 = 1 \), not to mention the problem of identifying the best choices of \( r \) and \( d \). In case the multiplicities \( m_i \) are all equal, it is not hard to show that optimal solutions (for given \( r \) and \( d \)) to this linear programming problem are given in parts (a) and (b) of the following corollary. These need not always be optimal if the coefficients are not all equal, so we consider in parts (c) and
Corollary IV.1.1.2: Let $Z = m_1p_1 + \cdots + m_mp_n$ for general points $p_i \in \mathbf{P}^2$ with $n \geq 1$ and $m_1 \geq \cdots \geq m_n \geq 0$, let $r \leq n$ and $d$ be positive integers and let $m$ be the mean of $m_1, \ldots, m_n$.

(a) If $r^2 \geq nd^2$, then $\alpha(Z) \geq mnd/r$.
(b) If $r^2 \leq nd^2$, then $\alpha(Z) \geq mr/d$.
(c) If $d^2 \geq r$, then $\alpha(Z) \geq (m_1 + \cdots + m_r)/d$.
(d) Assume $d^2 < r$ and let $j$ be an integer, $0 \leq j \leq d^2$.
   (i) If $j = 0$, then $\alpha(Z) \geq (m_1 + \cdots + m_d)/d$.
   (ii) If $j > 0$, let $t = \min\{r + (r - d^2)j/(r - d^2) + j, n\}$ and set $m_{t+1} = 0$ if $t = n$; then

$$\alpha(Z) \geq (1/d)\left(\frac{t - \lfloor t\rfloor} {r - d^2 + j}\right) + \sum_{1 \leq i \leq d^2 - j} m_i + \sum_{d^2 - j < i \leq t} \frac{jm_i} {r - d^2 + j}.$$ 

Sketch of proof: Each part of the corollary applies Theorem IV.1.1.1 for various values of the $a_i$. For (a), take $a_0 = r$ and $a_1 = \cdots = a_n = d^2$. For (b), take $a_0 = n$ and $a_i = r$, $i > 0$. For (c), take $a_i = 1$ for $i \leq r$ and $a_i = 0$ for $i > r$. For (d)(i), take $a_i = 1$ for $i \leq d^2$ and $a_i = 0$ for $i > d^2$. For (d)(ii), take $a_i = 1$ for $i \leq d^2 - j$ and $a_i = j/(r - d^2 + j)$ for $d^2 - j < i \leq \lfloor t\rfloor$. If $t = n$, then $m_{t+1} = 0$ (and so is $t - \lfloor t\rfloor$), but if $t < n$, then take $a_{t+1} = (t - \lfloor t\rfloor)j/(r - d^2 + 1)$.

One can formally verify that the values of the $a_i$ given in (d)(ii) satisfy the necessary conditions to apply Theorem IV.1.1.1, but it may be helpful to briefly discuss how these values come about. The idea giving rise to the values of $a_i$ in (d)(ii) is to find extremal sets (one set for each $j$) of values of the $a_i$, with the hope that for any given $Z$ one set will be close to an optimal solution that might be found by linear programming. By setting $a_0$ equal to 1 (a normalization we clearly can always do), we bound the values of the other $a_i$ above by 1. Since the multiplicities $m_i$ are nonincreasing, any optimal solution for the $a_i$ must also be nonincreasing. Intuitively, we would want to keep as many of the $a_i$ equal to 1 as possible. But in order to satisfy $d^2 \geq a_1 + \cdots + a_r$ we can keep at most the first $d^2$ of the $a_i$ equal to 1, in which case all of the other $a_i$ would have to be 0. Depending on the values of the $m_i$, however, we may be better off if we can make enough of the other $a_i$ positive. So, given $j$, we leave $a_1, \ldots, a_{d^2-j}$, alone, and spread $a_{d^2-j+1}, \ldots, a_{d^2}$, which are each 1 to start with, evenly over $a_{d^2-j+1}$ to $a_r$, which reduces $a_{d^2-j+1}, \ldots, a_{d^2}$ from 1 to $j/(r - d^2 + j)$, and raises $a_{d^2+1}, \ldots, a_r$ from 0 to $j/(r - d^2 + j)$, while keeping the condition $d^2 \geq a_1 + \cdots + a_r$ satisfied at equality. Now, although this may have worsened things (since we may well have reduced $a_1m_1 + \cdots + a_rm_r$), we can hope to more than make up for this since we can now increase some of the remaining $a_i$ from 0 (which they were before) to $j/(r - d^2 + j)$. How many of the $a_i$ which we can increase is limited by the condition $r \geq a_1 + \cdots + a_n = d^2 + a_{r+1} + \cdots + a_n$; moreover, because of fractional effects, the last $a_i$ which we can manage to increase from 0 might be limited to being increased only by a fraction of $j/(r - d^2 + j)$, which accounts for the anomalous behavior of $a_{t+1}$. ♦

The bounds given in Corollary IV.1.1.2 can be computed by running the scripts unifbounds or bounds. The script ezbalphad, which is called by bounds, checks all possible $r$, $d$ and $j$ from Corollary IV.1.1.2(d).
Additional constructions of nef divisors are given in [Bi] and [H10], with applications to multipoint Seshadri constants, which we now briefly discuss.

If $Z = p_1 + \cdots + p_n$ for $n$ generic points $p_i$, let us denote $\varepsilon(Z)$ by $\varepsilon(n)$. Since it is always true that $\varepsilon(n) \leq 1/\sqrt{n}$, one is interested in obtaining lower bounds on $\varepsilon(n)$. For example, Xu obtains $\varepsilon(n) \geq (1/\sqrt{n})\sqrt{1-1/n}$ (see Theorem 1(a), [Xu1]). Using the same basic method pushed a bit harder, Szemberg and Tutaj-Gasińska improve this for $n > 9$ to $\varepsilon(n) \geq (1/\sqrt{n})\sqrt{1-1/(n+1)}$ [ST], and by using in addition a monodromy argument, Szemberg obtains an even better result [Sz], a simplified version of which obtained by applying [Mi] is that $\varepsilon(n) \geq (1/\sqrt{n})\sqrt{1-1/(5n)}$ for $n > 9$.

However, [H10] takes a different approach, using constructions of nef divisors to obtain bounds. If $dE_0 - m(E_1 + \cdots + E_n)$ is a nef divisor on the blow up $X$ of the points $p_i$, it follows from the definition that $\varepsilon(n) \geq m/d$. Thus, for example, it follows from the proof of Corollary IV.1.1.2(a,b) that $\varepsilon(n) \geq \max\{\{d/r\}nd^2 \leq r^2 \leq n^2\} \cup \{r/(nd)|r \leq n, r^2 \leq nd^2\}$. By studying $\max\{\{d/r\}nd^2 \leq r^2 \leq n^2\} \cup \{r/(nd)|r \leq n, r^2 \leq nd^2\}$ it follows for any $a \geq 1$ that $\varepsilon(n) \geq (1/\sqrt{n})\sqrt{1-1/(an)}$ for almost all $n$ (in the following precise sense: if $T(a, n) = \{k \leq n|\varepsilon(k) \geq (1/\sqrt{k})\sqrt{1-1/(ak)}\}$, then $\lim_{n \to \infty}|T(a, n)|/n = 1$; see Proposition I.2(b)(iiii) of [H10]).

Similarly but by a different method, [Bi] also obtains bounds by constructing nef divisors involving rational approximations to $\sqrt{n}$ (but only for certain values of $n$ over the complex numbers); it may be of interest to mention that the methods of [H10] generalize to arbitrary characteristic and to arbitrary algebraic surfaces.

However, exact values of $\varepsilon(n)$ are known, all by work of Nagata, only for $n \leq 9$ (in which case $\varepsilon(n) = c_n/n$ for $c_n$ as given in the first paragraph of Section IV.1) and when $n$ is a square (in which case $\varepsilon(n) = 1/\sqrt{n}$).

IV.1.2. Bounds by unloading

As an alternative to Theorem IV.1.1.1, we can use a process that can conveniently be referred to as unloading. The idea is based on the fact that given a divisor class $D$ on a surface $X$ and some finite set $S$ of classes of effective, irreducible divisors, if for some $F \in S$ we have $F \cdot D < 0$, then clearly $D$ is the class of an effective divisor if and only if $D - F$ is. Unloading (in a sense that is slightly more general than its use in the literature) consists of checking $D \cdot F$ for each $F \in S$, and replacing $D$ by $D - F$ whenever $D \cdot F < 0$ and continuing with the new $D$. (For the classical notion of unloading, see pp. 425–438 of vol. 2 of [EC], where it is referred to as scaricamento, or see [DuV1].) Eventually, $D$ reduces to a class $D'$ such that either $D'$ is obviously not effective (because, perhaps, $D' \cdot E_0 < 0$ or $D' \cdot (E_0 - E_1) < 0$) or such that $D' \cdot F \geq 0$ for all $F \in S$.

With respect to the specialization used in the proof of Theorem IV.1.1.1, we can take $S$ to consist of the classes $[E_i - E_{i+1}]$ for $0 < i < n$, $[E_n]$ and $d[E_0] - [E_1 + \cdots + E_r]$, and we look for the largest $t$ such that $D = [tE_0 - (m_1E_1 + \cdots + m_nE_n)]$ unloads to a class $D'$ with $D' \cdot (E_0 - E_1) < 0$, in which case $t + 1$ is a lower bound for $\alpha(Z)$. I have included the script bhalpha to compute the bound obtained via unloading with respect to any chosen $r$ and $d$.

In the special case that $r \leq d^2$, then $d[E_0] - [E_1 + \cdots + E_r]$ is nef. It is not hard to then see that the result of the unloading process is the same as just testing against this nef
divisor, hence, assuming that \( m_1 \geq \cdots \geq m_n \), we get the bound \( \alpha(m_1 p_1 + \cdots + m_n p_n) \geq (m_1 + \cdots + m_r)/d \). One can also give a formula for the result of this unloading process in another extremal case, \( 2r \geq n + d^2 \). In this case, for \( Z = m(p_1 + \cdots + p_n) \), we have \( \alpha(Z) \geq 1 + ud + \min\{d - 1, [\rho/d] - 1\} \), where \( u \geq 0 \) and \( \rho \) are defined by \( mn = ur + \rho \), with \( 0 < \rho \leq r \).

The idea of bounding \( \alpha \) using unloadings and specializations to infinitely near points is due to Roé [R1], who actually uses a sequence of increasingly special specializations of infinitely near points, applying unloading after each specialization. Roé uses a sequence of \( n - 2 \) specializations, corresponding to sets \( S_i \), \( 3 \leq i \leq n \), of classes of reduced irreducible divisors, where \( S_i = \{[E_n]\} \cup \{[E_j - E_{j+1}] : 1 < j < n\} \cup \{[E_1 - E_2 - \cdots - E_i]\} \). Starting with \( F_t = tE_0 - m_1E_1 - m_2E_2 - \cdots - m_nE_n \), Roé’s algorithm consists of unloading \( F_t \) with respect to \( S_3 \) to get \( F_t^{(3)} \), then unloading \( F_t^{(3)} \) with respect to \( S_4 \) to get \( F_t^{(4)} \), etc., eventually ending up with \( F_t^{(n)} = tE_0 - m_1^{(n)}E_1 - m_2^{(n)}E_2 - \cdots - m_n^{(n)}E_n \). Roé’s bound is then \( \alpha(Z) \geq m_1^{(n)} \), which comes from the fact that \( [F_t^{(n)}] \) and hence \( F_t \) cannot be classes of effective divisors unless \( t \geq m_1^{(n)} \). This bound can be computed with the scripts \texttt{unifroealpha} and \texttt{roealpha}.

Although it is hard to give a simple formula for the exact value of the result of this method, an asymptotic analysis by Roé [R1] shows that his unloading procedure gives a lower bound for \( \alpha(Z) \) which is always better than \( m(\sqrt{n - 1} - \pi/8) \), for \( Z = m(p_1 + \cdots + p_n) \) with \( n > 2 \) general points. It should be noted however that this formula often substantially understates the result of the full algorithm.

**IV.1.3. Bounds by a modified unloading**

Assume a specialization as in the second paragraph of Section IV.1.2; we may assume \( d[E_0] - [E_1 + \cdots + E_r] \) is the class of a smooth curve \( C \). Unloading \( F = [tE_0 - m_1E_1 - \cdots - m_nE_n] \) with respect to the set \( S \) of Section IV.1.2 uses the fact that, if \( C \cdot F < 0 \), then \( F - [C] \) is the class of an effective divisor if and only if \( F \) is. However, the requirement \( C \cdot F < 0 \) can be relaxed, since all we really need is \( h^0(C, \mathcal{O}_C(F)) = 0 \) in order to ensure that \( F - [C] \) is the class of an effective divisor if and only if \( F \) is. By joint work with J. Roé [HR], using the notion of a flex of a linear series on \( C \), one can show (in characteristic 0) that \( h^0(C, \mathcal{O}_C(F)) = 0 \) if either \( t < d \) and \( (t + 1)(t + 2)/2 \leq m_1 + \cdots + m_r \), or \( t > d - 3 \) and \( F \cdot C \leq g - 1 \), where \( g = (d - 1)(d - 2)/2 \). (Recall that \( C \) is the proper transform of a plane curve \( C' \). The idea is to choose \( p_1 \in C' \) so that it is not a flex for the complete linear series associated to the restrictions to \( C \) of the divisors occurring during the unloading process. This is automatic in characteristic 0 as long as \( p_1 \) is a general point of \( C' \), but in positive characteristics every point of \( C' \) may be a flex for a given, even complete, linear series [Ho]). Using this test in place of the more stringent test \( C \cdot F < 0 \) discussed at the beginning of Section IV.1.2 gives what may be called the modified unloading procedure. Since this modified procedure uses a less stringent test, a larger (or at least as large) degree is needed to pass the test, so it gives bounds on \( \alpha \) which are at least as good as the original unloading procedure. These new bounds can be computed by running \texttt{HRalpha} or \texttt{unifHRalpha}.

Although this modified unloading procedure is somewhat difficult to analyze in general, in two extremal cases Roé and I [HR] derive the following simple bounds for
\(Z = m(p_1 + \cdots + p_n)\), for which define \(u \geq 0\), \(\rho\) and \(s\) by requiring \(mn = ur + \rho\), with \(0 < \rho \leq r\), where \(s\) is the largest integer such that \((s+1)(s+2) \leq 2\rho\) and \(s < d\):

- if \(2n \geq 2r \geq n + d^2\) (and hence \(r^2 \geq nd^2\)), then \(\alpha(Z) \geq s + ud + 1\). (The bound given by this formula can be computed by running **ezunifHRalpha**.)
- if \(rd(d+1)/2 \leq r^2 \leq \min\{n^2, nd^2\}\), then \(\alpha(n; m) \geq 1 + \min\{(mr + g - 1)/d, s + ud\}\). (The bound given by this formula can be computed by running **ezunifHRalphab**.)

To apply these bounds, one must choose \(r\) and \(d\). For \(d = \lfloor \sqrt{n} \rfloor\) and \(r = \lfloor (n + d^2)/2 \rfloor\) (in which case the first bound applies when \(n + d^2\) is even and the second when \(n + d^2\) is odd), Figure 3 shows the values \(10 \leq n \leq 480\) and \(1 \leq m \leq 400\) that these bounds imply Nagata’s conjectural bound \(\alpha(Z) \geq m \sqrt{n}\); there are 131261 such points \((n, m)\), or 69.8%. It is also worth considering \(d = \lfloor \sqrt{n} \rfloor\) and \(r = \lfloor d \sqrt{n} \rfloor\), in which case the second bound above easily leads to:

**Corollary IV.1.3.1:** Let \(Z = m(p_1 + \cdots + p_n)\) with \(1 \leq m \leq (n - 5\sqrt{n})/2\) for \(n \geq 10\) general points \(p_i\) of \(\mathbb{P}^2\); then \(\alpha(Z) \geq m \sqrt{n}\).

This is a substantial improvement on previous such results (see [Ev1] and [HR]).

### IV.1.4. Bounds using \(\Psi\)

The subsemigroup \(\Psi \subset \text{Cl}(X)\), introduced in Section III.2, contains the subsemigroup of classes of effective divisors. Thus, given \(Z\), the least \(t\) such that \(F_t(Z) \in \Psi\) is a lower bound for \(\alpha(Z)\). This sometimes gives an optimal bound. For example, if \(Z = 90p_1 + 80p_2 + 70p_3 + 60p_4 + 50p_5 + 40(p_6 + p_7 + p_8) + 30p_9 + 20p_{10} + 10p_{11}\), then the least \(t\) such that \(F_t(Z)\) is in \(\Psi\) is 179, hence in fact \(\alpha(Z) = 179\), since \(e(F_{179}(Z)) > 0\). Finding the least \(t\) such that \(F_t(Z) \in \Psi\) is somewhat tedious, so I have provided the script **Psibound** for doing so.

### IV.1.5. Comparisons

For subschemes \(Z\) whose multiplicities are not too uniform, the lower bound on \(\alpha(Z)\) given by testing against \(\Psi\) can be the best, as it is for \(Z = 90p_1 + 80p_2 + 70p_3 + 60p_4 + 50p_5 + 40(p_6 + p_7 + p_8) + 30p_9 + 20p_{10} + 10p_{11}\) (see Section IV.1.4). For example, Roé’s method [R1] of unloading gives \(\alpha(Z) \geq 162\), and the best result achievable using Corollary IV.1.1.2 turns out to be \(\alpha(Z) \geq 173\), whereas testing against \(\Psi\) gives \(\alpha(Z) \geq 179\) (and hence \(\alpha(Z) = 179\) as discussed above).

However, if the multiplicities are fairly uniform, testing against \(\Psi\) does not give a very good bound. For example, for \(Z = m(p_1 + \cdots + p_n)\) with \(n > 9\), it is easy to see that \(F_t(Z) \in \Psi\) for all \(t \geq 3m\), so testing against \(\Psi\) gives the bound \(\alpha(Z) \geq 3m\). This compares poorly with bounds via the other methods, which are typically very close to, but usually less than, \(m \sqrt{n}\). (Except when \(n\) is a square, currently only the unloading method of [R1] and the modified unloading method, discussed in Section IV.1.2 and Section IV.1.3, resp., ever are substantially better than \(m \sqrt{n}\), and even these only when \(m\) is not too large compared to \(n\).)

Thus for uniform subschemes \(Z = m(p_1 + \cdots + p_n)\) one is better off using some method other than testing against \(\Psi\), such as testing against nef divisors, as discussed in...
Graph showing all \((n, m)\) such that the Harbourne-Roé bounds, using \(d = \lfloor \sqrt{n} \rfloor\) and \(r = \lfloor (n + d^2)/2 \rfloor\), imply Nagata’s conjectural bound \(\alpha(Z) \geq m \sqrt{n}\).

Figure 3
Section IV.1.1. In this case one has, for any $d$ and $r$, easy to implement tests, as given in Corollary IV.1.1.2. By comparison, the result of unloading with respect to the divisor $C = dE_0 - (E_1 + \cdots + E_r)$, as discussed in Section IV.1.2, is, except in certain special cases, harder to compute since there is not always a simple formula for the result. Since one rarely gets something for free, it is not surprising, for given $r$ and $d$, that the bounds given by testing against a nef divisor are never better than those given by unloading.

To see this, let $Z = m(p_1 + \cdots + p_n)$, and assume $\alpha(Z) \geq t$ is the bound given by unloading with respect to $C$. Also, with respect to the same $r$ and $d$, let $F = a_0E_0 - (a_1E_1 + \cdots + a_mE_n)$ be the nef test class in the proof of Theorem IV.1.1.1. The unloading method unloads a divisor $D = tE_0 - m(E_1 + \cdots + E_n)$ to a divisor $D'$ which meets $C$, $E_n$ and $E_i - E_{i+1}$, for all $i$, nonnegatively. But $D - D'$ is a sum of multiples of these same divisors, which are all (linearly equivalent to) effective divisors, so each meets $F$ nonnegatively. In addition, $F$ is a sum of these same divisors, each of which $D'$ meets nonnegatively, so $D' \cdot F \geq 0$ too. Thus $D \cdot F = 0$, which shows that testing against the nef divisor can never rule out the candidate obtained by unloading.

Moreover, if $r^2 > nd^2$, unloading can definitely be better. For example, take $n = 22$ and $m = 3$. Then the best choice of $r$ and $d$ with $r^2 \geq nd^2$ is $r = 19$ and $d = 4$, while the best choice of $r$ and $d$ with $r^2 \leq nd^2$ is $r = 14$ and $d = 3$. Using Corollary IV.1.1.2(a,b) with either choice of $r$ and $d$ gives $\alpha \geq 14$, but unloading with respect to $r = 19$ and $d = 4$ gives $\alpha \geq 15$. Since Corollary IV.1.1.2 is optimal in this case, we see unloading sometimes gives a better result than can be obtained by any use of Theorem IV.1.1.

On the other hand, for $Z = m(p_1 + \cdots + p_n)$ with $r^2 \leq nd^2$ and $r \leq n$, the bound $\alpha(Z) \geq mr/d$ obtained by testing against a nef divisor, cannot be improved by unloading with respect to $dE_0 - (E_1 + \cdots + E_n)$, and hence unloading and testing against a nef divisor give the same result in these circumstances. (This is because for unloading to give a better bound, the class $D = [(mr/d)E_0 - m(E_1 + \cdots + E_n)]$ would have to unload to something obviously not effective, but unloading cannot get started unless $D$ meets $dE_0 - E_1 - \cdots - E_r$ nonnegatively, which it does not.) But as the example of the preceding paragraph shows, if $r^2 \leq nd^2$, although one cannot do better than $mr/d$ by unloading with respect to $r$ and $d$, one can still hope to do better than $mr/d$ by unloading using some choices $r'$ and $d'$ in place of $r$ and $d$.

Since the modified unloading procedure of Section IV.1.3 uses a less stringent test than does unloading, as in Section IV.1.2, with respect to $C = dE_0 - (E_1 + \cdots + E_r)$ (in the sense that in order to be allowed to subtract $C$ and continue the unloading process, for the former the intersection with $C$ can in most cases be as much as $g - 1$, where $g$ is the genus of $C$, whereas for the latter the intersection must be negative), we see that bounds obtained via the latter method can never be better than those obtained by the former. The advantage of the latter method is that no hypotheses are required on the characteristic.

There is also Roé’s unloading method [R1], discussed in Section IV.1.2. As shown in [H9], for $m$ sufficiently large compared to $n$, the results of Corollary IV.1.1.2 are always better than Roé’s unloading method. However, when $m$ is not too large compared with $n$, examples indicate that Roé’s method gives the best bounds currently known. Consider, for instance, two examples using modified unloading (Section IV.1.3). For $n = 1000$ and $m = 13$, Roé’s method gives $\alpha \geq 421$, whereas modified unloading, using $r = 981$ and $d = 31$, gives $\alpha \geq 424$, and the SHGH conjectural value of $\alpha$ is 426. For $n = 9000$ and
$m = 13$, things become reversed: Roé’s method gives $\alpha \geq 1274$, while modified unloading using $r = 8918$ and $d = 94$ gives only $\alpha \geq 1267$; the SHGH conjectural value of $\alpha$ in this case is $1279$.

An interesting feature of these examples is that in both cases the bounds are better than $\lceil m \sqrt{n} \rceil + 1$, conjectured by Nagata (Conjecture III.1.1): For $m = 13$ and $n = 1000$, we have $\lceil m \sqrt{n} \rceil + 1 = 412$, while for $m = 13$ and $n = 9000$, we have $\lceil m \sqrt{n} \rceil + 1 = 1234$. Indeed, whereas most known lower bounds for $\alpha(Z)$ for $Z = m(p_1 + \cdots + p_n)$ are less than $\sqrt{n}$ (since after all Nagata’s conjecture is still open), the method of [R1] and that of modified unloading are among the few that in certain situations gives bounds that can be substantially better than $m \sqrt{n}$. In particular, if $m$ is no bigger than about $\sqrt{n}$, the method of [R1] consistently (and probably always, although this looks hard to prove) gives a lower bound that is at least as big as $m \sqrt{n}$, and gets better as $m$ decreases until, for $m = 1$ it is easy to show that it gives the actual value of $\alpha(Z)$. If one chooses $r$ and $d$ carefully (depending on $n$), examples indicate that the modified unloading procedure does nearly as well as the method of [R1] when $m$ is small compared to $n$, and is substantially better for larger $m$. The method of [R1], of course, has the advantage of being characteristic free and does not depend on careful choices of other parameters. The modified unloading method, on the other hand, sometimes gives a lower bound which is equal to the SHGH conjectural value (which is known to be an upper bound), and thus determines $\alpha$ exactly (as happens, for example, when $n = d(d+1)$ and $m = d + 1$ for $d > 2$ even, as discussed in Section II.3, or $n = 38$ with $m = 200$, as mentioned in Section III.4).

Thus, in terms of getting the best bound for a given $Z$, the modified unloading method (at least in characteristic 0) is often the best. It has, compared with methods (such as Corollary IV.1.1.2) which test against nef divisors, the disadvantage of being harder to compute, unless special values for $r$ and $d$ are chosen for which a formula applies. But since Corollary IV.1.1.2 works for essentially any $r$ and $d$, sometimes one can do better by applying Corollary IV.1.1.2 than one can by applying the formula of Section IV.1.3 where one’s choices of $r$ and $d$ are more restricted.

This raises the question of which $r$ and $d$ give the best result when applying Corollary IV.1.1.2(a, b). In case (a), $n \geq r$ and $r^2 \geq nd^2$ imply $n \geq d^2$ (and even $r \geq d^2$), while in case (b), having $r^2 \leq nd^2$ and $r \leq n$ but trying to maximize $r/d$ shows that it is enough to consider values of $d$ with $d \leq \lceil \sqrt{n} \rceil$. In short, in cases (a) and (b), we may as well only consider $d$ with $d \leq \lceil \sqrt{n} \rceil$. Moreover, given such a $d$, the best choice of $r$ is evidently $\lceil d \sqrt{n} \rceil$ for case (a) and $\lfloor d \sqrt{n} \rfloor$ for case (b). It is still (as far as I can see) not easy to tell which $d$ is best without checking each $d$ from 1 to $\lceil \sqrt{n} \rceil$, hence I have included the script bestdra for case (a), and bestdrb for case (b), to do just that. Alternatively, $d = \lfloor \sqrt{n} \rfloor$ often seems to be a good choice. For this choice of $d$ and the corresponding optimal choices of $r$, (a) ends up giving a better bound than (b) if $n - d^2$ is even, while (b) is better if $n - d^2$ is odd.

IV.2. Bounds on $\tau$

In some ways, $\tau$ is easier to compute than $\alpha$. For example, given $Z = m(p_1 + \cdots + p_n)$ for $n \geq 9$ general points, [HHF] proves by an easy specialization argument that

$$\tau(Z) \leq m \lceil \sqrt{n} \rceil + \lceil (\lceil \sqrt{n} \rceil - 3)/2 \rceil.$$
If \( n \geq 9 \) is a square and \( m > (\sqrt{n} - 2)/4 \), it follows (see [HFF]) in fact that

\[
\tau(Z) = m\sqrt{n} + \lceil(\sqrt{n} - 3)/2\rceil.
\]

Thus \( \tau \) is known in some situations where \( \alpha \) is only conjectured.

Moreover, via an observation of Z. Ran, bounds on \( \alpha \) give rise to bounds on \( \tau \). In particular, given \( Z = m(p_1 + \cdots + p_n) \) with \( p_i \) general, if \( \alpha(Z) \geq c_n m \) for all \( m \) (where \( c_n > 0 \) depends only on \( n \)), then

\[
\tau(Z) \leq -3 + \lceil (m + 1) \max\{\sqrt{n}, n/c\} \rceil
\]

(see Remark 5.2 of [H9]). Thus, for example, the bounds of Corollary IV.1.1.2(a, b) lead to bounds on \( \tau \).

It should not be surprising that \( \tau \) might be easier to handle than \( \alpha \). Being always able to compute \( \alpha(Z) \) is equivalent to being always able to compute \( h_Z \) and hence \( \tau(Z) \), while the reverse does not seem to be true. Moreover, arguments typically involve specializations. One can hope to compute \( \tau \) exactly using a specialization that drops \( \alpha \) (and thereby gives us something to work with) while leaving \( \tau \) unchanged, but this of course will not work to compute \( \alpha \), only to give a lower bound.

The scripts \texttt{findtau} and \texttt{uniffindtau} give lower bounds for \( \tau \) which via the SHGH Conjecture are expected to be the actual values. Thus most interest is in finding upper bounds on \( \tau \), and indeed, quite a few upper bounds have been given, both on \( \mathbf{P}^2 \) and in higher dimensions (see, for example, [FL], for various results and additional references).

Given \( Z = m_1p_1 + \cdots + m_np_n \), bounding \( \tau(Z) \) is mostly of interest for \( n > 9 \) since for \( n \leq 9 \), for any disposition of the points, the Hilbert function of \( I(Z) \) (and hence \( \tau(Z) \)) is known (see [H4] for \( n \leq 8 \) or [H5]). For \( n > 9 \), the results of [H5] also allow one to compute \( \tau(Z) \) exactly, if the points \( p_i \) lie on a plane cubic. If the points \( p_i \) are general, and \( t \) is the value of \( \tau(Z') \) (computed via [H5]) for some specialization \( Z' \) of the points \( p_i \) to a plane cubic, then by semicontinuity \( \tau(Z) \leq t \). For \( Z = m(p_1 + \cdots + p_n) \) with \( n > 9 \), this gives the bound

\[
\tau(Z) \leq mn/3.
\]

This bound is similar in concept to but better than a bound given by Segre [Seg], obtained by specializing to a conic, which for \( Z = m(p_1 + \cdots + p_n) \) with \( n > 9 \) gives only

\[
\tau(Z) \leq mn/2.
\]

Improved bounds for \( \tau(Z) \) for \( Z = m_1p_1 + \cdots + m_np_n \) with \( p_i \) general are given by Catalisano [Cat1], Gimigliano [Gi3] and Hirschowitz [Hi2]. For \( Z = m_1p_1 + \cdots + m_np_n \) with general points \( p_i \) and \( m_1 \geq \cdots \geq m_n \geq 0 \), Gimigliano’s result is that

\[
\tau(Z) \leq m_1 + \cdots + m_d
\]

as long as \( d(d + 3)/2 \geq n \), while Hirschowitz’s result is that \( \tau(Z) \leq d \) if

\[
\lceil (d + 3)/2 \rceil \lceil (d + 2)/2 \rceil > \sum_i m_i(m_i + 1)/2.
\]
Catalisano’s result is somewhat complicated, but generalizes and often improves Gimigliano’s. For $Z = m(p_1 + \cdots + p_n)$ with $n > 9$ these all show that $\tau(Z)$ is at most approximately $m\sqrt{2n}$. For $n$ sufficiently large, this clearly is better than $\tau(Z) \leq mn/3$.

The bound $\tau(Z) \leq m[\sqrt{n}] + \lceil(\sqrt{n} - 3)/2\rceil$, mentioned above ([HHF]), results from specializing $n > 9$ points to a smooth curve of degree $[\sqrt{n}]$. Two other bounds which are also on the order of $m\sqrt{n}$ are Ballico’s [B] for which $\tau(Z) \leq d$ if

$$d(d + 3) - nm(m + 1) \geq 2d(m - 1) - 2$$

(but note that $\tau(Z) \leq m[\sqrt{n}] + \lceil(\sqrt{n} - 3)/2\rceil$ is better for any given $n$ if $m$ is large enough) and Xu’s for which $\tau(Z) \leq d$ if

$$3(d + 3) > (m + 1)\sqrt{10n}$$

(although $\tau(Z) \leq m[\sqrt{n}] + \lceil(\sqrt{n} - 3)/2\rceil$ is better if $n$ is sufficiently large).

By employing a sequence of specializations to infinitely near points similar to what he did for bounding $\alpha$, Roé [R2] obtains an upper bound on $\tau$. The method applies for any $Z = m_1p_1 + \cdots + m_np_n$, with $p_i$ general and $n \geq 2$. For $Z = m(p_1 + \cdots + p_n)$, [R2] denotes this upper bound by $d_1(m,n)$ and proves

$$d_1(m,n)+1 \leq m(n/(n-1))(\Pi_{i=2}^{n-1}((n-1+i^2)/(n-1+i^2-i))) \leq (m+1)(\sqrt{n}+1.9+\pi/8).$$

The bound $\tau(Z) \leq (m + 1)(\sqrt{n + 1.9 + \pi/8}) - 1$ compares very well with the bound $\tau(Z) \leq m[\sqrt{n}] + \lceil(\sqrt{n} - 3)/2\rceil$: the former is better for approximately 60% of the values of $n$ between any two successive squares.

Given a curve $C$, the idea of Roé’s algorithm is that for any $F$, by taking cohomology of $0 \rightarrow \mathcal{O}_X(F - C) \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_C \otimes \mathcal{O}_X(F) \rightarrow 0$, we have $h^1(X, \mathcal{O}_X(F)) = 0$ if $h^1(X, \mathcal{O}_X(F - C)) = 0$ and $h^1(X, \mathcal{O}_C \otimes \mathcal{O}_X(F)) = 0$. In Roé’s case, $C$ is always rational so $h^1(X, \mathcal{O}_C \otimes \mathcal{O}_X(F)) = 0$ is guaranteed if $F \cdot C > -2$, and he handles $h^1(X, \mathcal{O}_X(F - C)) = 0$ by induction.

In somewhat more detail, start with $Z = m_1p_1 + \cdots + m_np_n$ with $p_i$ general, and so we may assume $m_i \geq m_{i+1} \geq 0$ for all $i$. We have the corresponding divisor class

$$F = [tE_0 - m_1E_1 - \cdots - m_nE_n]$$

where $t$ is as yet undetermined. Now specialize so that each element of $S = \{[E_i - E_{i+1}] : 1 < i < n\} \cup \{[E_n]\}$ and $[E_1 - E_2]$ is the class of a reduced, irreducible divisor. Now, $F \cdot [E_1 - E_2] \geq -1$ is certainly true to start with (in fact, we have $F \cdot [E_1 - E_2] \geq 0$). If $F \cdot [E_1 - E_2 - E_3] \geq -1$, then fine, but otherwise replace $F$ by $F - [E_1 - E_2]$ and unload the result with respect to $S$, and continue replacing and unloading in the same way until $F \cdot [E_1 - E_2 - E_3] \geq -1$. Note that throughout this sequence of operations we have $F \cdot [E_1 - E_2] \geq -1$, so (taking $[C] = [E_1 - E_2]$) we have $h^1(X, \mathcal{O}_C \otimes \mathcal{O}_X(F)) = 0$. Also, unloading involves a succession of replacements of $F$ by $F - [E]$, where $[E]$ is always either $[E_i - E_{i+1}]$ for some $i$ or $[E_n]$, and can always be carried out in such a way that at each step we have $F \cdot [E] > -2$. Thus we always have $h^1(E, \mathcal{O}_E \otimes \mathcal{O}_X(F)) = 0$, where $E$ is a curve whose class is, at various times, $[E_i - E_{i+1}]$ for some $i$ or $[E_n]$.

So eventually $F$ turns into a class for which $F \cdot [E_1 - E_2 - E_3] \geq -1$, $F \cdot [E_i - E_{i+1}] \geq 0$ for all $i$ and $F \cdot E_n \geq 0$. We now further specialize so that $[E_1 - E_2 - E_3]$ is the class of an
irreducible divisor, and keep replacing \( F \) by \( F - [E_1 - E_2 - E_3] \), unloading with respect to \( S \) after each replacement, as long as \( F \cdot [E_1 - E_2 - E_3 - E_4] < -1 \). We continue in this way, specializing successively so that each \([E_1 - E_2 - \cdots - E_i]\) in turn becomes the class of an irreducible divisor, and replacing \( F \) by \( F - [E_1 - \cdots - E_i] \) and unloading with respect to \( S \) after each replacement, as long as \( F \cdot [E_1 - E_2 - \cdots - E_i+1] < -1 \). Eventually we end up with a class \( F' = [tE_0 - m_1'E_1 - \cdots - m_n'E_n] \) with \( m_i \geq m_{i+1} \geq 0 \) for all \( i \). We want to choose \( t \) to be large enough to start with so that we can keep subtracting \([dE_0 - (E_1 + \cdots + E_r)]\) and unloading with respect to \( S \) until we eventually obtain a class \( F' = t'[E_0] \) for some \( t' \), while along the way always keeping \( h^1(D, \mathcal{O}_D \otimes \mathcal{O}_X(F)) = 0 \). The latter is guaranteed (in characteristic 0) if both \( F \cdot E_0 \geq d - 2 \) and \( F \cdot D \geq g - 1 \), where \( g = (d - 1)(d - 2)/2 \) is the genus of \( D \).

The output of the algorithm of the previous paragraph is easy but tedious to compute in any given case; to get a nice formula we seem to need to choose \( r \) and \( d \) carefully. For example, let \( Z = m(p_1 + \cdots + p_n) \) with \( p_i \) general in characteristic 0. Assume \( r \leq n \) and define \( u \geq 0 \) and \( 0 < \rho \leq r \) via \( mn = ur + \rho \). If \( 2r \geq n + d^2 \) (such as is the case for \( d = \lceil \sqrt{n} \rceil \) and \( r = \lceil d\sqrt{n} \rceil \)), the algorithm gives

\[
\tau(Z) \leq \max\{[(mr + g - 1)/d], (u + 1)d - 2\},
\]

while if \( r \leq d^2 \), then the algorithm gives

\[
\tau(Z) \leq \max\{[(\rho + g - 1)/d] + ud, (u + 1)d - 2\}.
\]

Using \( d = \lceil \sqrt{n} \rceil \) and \( r = n \), the latter formula gives a bound which is always at least as good as that mentioned above from [HHF]. And when \( m \) is sufficiently large, the former formula becomes \( \tau(Z) \leq [mr/d + (d - 3)/2] \), which for a given \( n \) with \( m \) sufficiently large, gives a better bound than the bound \( d_1(m, n) \) given in [R2]. (To justify this claim, note that by a method similar to how [R2] shows that \( d_1(m, n) \leq -1 + m(n/(n-1))(\Pi_{i=2}^{n-1}((n-1+i^2)/(n-1+i^2-i))) \) one can show that \( m(n/(n-1))(\Pi_{i=2}^{n-1}((n-1+i^2)/(n-1+i^2-i))) - \sum_{i=3}^{n} n/(i(n-1) + i(i - 1)(i - 2)) \leq d_1(m, n) \). But \( (n/(n-1))(\Pi_{i=2}^{n}((n - 1 + i^2)/(n - 1 + i^2 - i))) \geq n/(\sqrt{n - 3/\pi} + d/\sqrt{n - 1}); \) see the proof of Proposition 4.2 of [H9]. The claim now follows for \( m \) large enough from the fact that \( n/(\sqrt{n - 3/\pi} + 1) > r/d \) for \( d = \lceil \sqrt{n} \rceil \) and \( r = \lceil d\sqrt{n} \rceil \) when \( n \geq 10 \).

The formulas

\[
\tau(Z) \leq \max\{[(mr + g - 1)/d], (u + 1)d - 2\}
\]

and

\[
\tau(Z) \leq \max\{[(\rho + g - 1)/d] + ud, (u + 1)d - 2\}
\]
can be evaluated by running `ezunifHRtau` and `ezunifHRtauB`, respectively. Since the algorithm works for any \( r \leq n \) and \( d \), it can sometimes do better than the formulas, which only work for certain values of \( r \) and \( d \). Thus I have provided scripts `unifHRtau` and `HRtau` to compute the output of the full algorithm with respect to any specified choice of \( r \leq n \) and \( d \).

V. Scripts

We close this survey with a collection of MACAULAY 2 scripts for computing some of the quantities and bounds discussed above. This is a verbatim listing: There are no \( \text{T\AE}X \) control sequences interspersed in the text of the scripts in the \( \text{T\AE}X \)file for this paper, so one can simply copy the text of the scripts from the \( \text{T\AE}X \)file directly into a file called (say) `BHscripts`. To run a script, such as `findres` (which computes a resolution of \( I(Z) \) for \( Z = m_1p_1 + \cdots + m_8p_8 \), where the \( p_i \) are assumed to be general and each \( m_i \) is an integer), start MACAULAY 2 and enter the command `load "BHscripts"`. Then enter the command `findres({m_1,...,m_8})`.

The required format for each script’s input parameters are described below, just before the listing for each script. Individual scripts can be run without loading the entire file, but many scripts defined below call one or more of the others, so be sure to load all scripts called by the one you wish to run.

```
-- These routines have been debugged on MACAULAY 2, version 0.8.52
-- Brian Harbourne, October 12, 2000
-- (July 30, 2001: revised scripts unifbounds, ezunifHRalpha and ezunifHRalphaB)

-- findres: This computes the syzygy modules in any resolution
-- of the saturated homogeneous ideal defining any eight or fewer general
-- fat points of P2. The hilbert function of the ideal is also found.
-- Call it as findres({m_1,...,m_8}) for n <= eight integers m_i.
-- Note that findres does not rely on Grobner bases, so it is fast by comparison.

findres = (l) -> (
if #l>8 then (<< "This script works only for up to 8 points." << endl;
<< "Please try again with an input list of at most 8 integers." << endl)
else (i:=0;
myflag2:=0;
w2:=
dd1:=0;
myker:=0;
ww:=
ww:=
-- the list l of multiplicities is, for simplicity, extended if need be
-- so that it has 8 elements.
while(#ww < 8) do ww=join(ww,{0});
n:=#ww;
n+=1;
ww=zz(ww); -- zero out negative elements of the list
al:=findalpha(ww); -- find alpha, the least degree t such that I_t \ne 0
d1:=al-2;
tau:=findtau(ww);
v4:=

-- list of number of syzygies in each degree t listed in v0
v3:=

-- list of dim of coker of \( \mu_t \) in each degree t listed in v0
```
Survey of Fat Points on $\mathbb{P}^2$

v2:={}; -- list of dim of ker of $\mu_t$ in each degree $t$ listed in v0
v1:={}; -- list of Hilbert function values for each degree listed in v0
v0:={}; -- list of degrees from alpha-2 to tau+2, where tau is the least
-- degree such that the fat points impose independent conditions
while (d1 <= tau+2) do ( -- loop from alpha-2 to tau+2, computing v0, v1 and v2
-- append the current degree $d_1$ to the list of degrees
v0=join(v0,{d1});
-- append the value of the hilbert function in degree $d_1$
-- to the list of values of the hilbert function
v1=join(v1,{homcompdim(fundom({d1,ww}))});
-- now compute and append to the list v2 the dimension myker of the
-- kernel of the map $\mu_t : I_t \otimes k[\mathbb{P}^2]_1 \to I_{t+1}$ where $t=d_1$
-- and I is the ideal of the fat points subscheme
if d1<a1 then (v2=join(v2,0)); -- d1<a1 means $I_{d1}=0$ so myker=0
if d1>=a1 then ( -- for d1>=a1, compute myker
myflag2=0;
w2=ww;
dd1=d1;
while(myflag2=0) do ( -- this loop implements the main theorem of [FHH]
-- which gives an algorithm for computing myker
w2=zr(w2);
w2=prmt(w2);
if homcompdim({dd1,w2})==0 then myflag2=1 else ( if dd1*6 - (dot(w2,{3,2,2,2,2,2,2})) <= 2 then ( dd1=dd1-6; w2 = {(w2#0)-3,(w2#1)-2,(w2#2)-2,(w2#3)-2,(w2#4)-2, (w2#5)-2,(w2#6)-2,(w2#7)-2} else ( if dd1*5 - (dot(w2,{2,2,2,2,2,2,1,1})) <= 1 then ( dd1=dd1-5; w2 = {(w2#0)-2,(w2#1)-2,(w2#2)-2,(w2#3)-2,(w2#4)-2, (w2#5)-2,(w2#6)-1,(w2#7)-1} else ( if dd1*4 - (dot(w2,{2,2,2,1,1,1,1,1})) <= 1 then ( dd1=dd1-4;
w2 = {(w2#0)-2,(w2#1)-2,(w2#2)-2,(w2#3)-1,(w2#4)-1, (w2#5)-1,(w2#6)-1,(w2#7)-1} else ( if dd1*3 - (dot(w2,{2,1,1,1,1,1,1,0})) <= 0 then ( dd1=dd1-3;
w2 = {(w2#0)-2,(w2#1)-1,(w2#2)-1,(w2#3)-1,(w2#4)-1, (w2#5)-1,(w2#6)-1,(w2#7)-1} else ( if dd1*2 - (dot(w2,{1,1,1,1,1,0,0,0})) <= 0 then ( dd1=dd1-2;
w2 = {(w2#0)-1,(w2#1)-1,(w2#2)-1,(w2#3)-1,(w2#4)-1, (w2#5),(w2#6),(w2#7))} else ( if dd1 - (dot(w2,{1,1,0,0,0,0,0,0}))<0 then ( dd1=dd1-1;
w2 = {(w2#0)-1,(w2#1)-1,(w2#2),(w2#3),(w2#4), (w2#5),(w2#6),(w2#7))} else ( myflag2=2));)));));)
if myflag2==1 then myker=0 else ( if dd1 - (dot(w2,{1,1,1,1,0,0,0,0})) == 0 then myker=homcompdim({dd1-1,ww});)

homcompdim({dd1-1,ww});

if {8*(w2#7)+3,www}=={dd1,w2} then myker=(w2#7)+1 else ( if homcompdim({dd1+1,w2})>3*(homcompdim({dd1,w2})) then myker=0 else myker=3*(homcompdim({dd1,w2})-homcompdim({dd1+1,w2})�);)

v2=join(v2,myker));
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```plaintext
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d1=d1+1;
scan(#v0, i->( -- this scan computes v3 from v2 and v1
  if i<2 then v3=join(v3,{0}) else 
    if v2#(i-1)>-1 then v3=join(v3,{v1#(i-3)*v1#(i-1)+v2#(i-1)}) else 
      if v1#(i-3)*v1#(i-1)>=0 then 
        v3=join(v3,{v1#(i-3)*(v1#(i-1))}) else 
      v3=join(v3,{v1#(i-3)*(v1#(i-1))})) else 
  v3=join(v3,{0}));

scan(#v0, i->( -- this scan computes v4 from v3 and v1
  if i<3 then v4=join(v4,{0}) else 
    v4=join(v4,{v3#i-v1#i+3*(v1#(i-1))-3*(v1#(i-2))+v1#(i-3)}));

<< "The output matrix has four columns. Column 1 indicates" << endl;
<< "each degree from alpha-2 (where alpha is the least" << endl;
<< "degree t such that I_t > 0 for the fat points ideal I)" << endl;
<< "to tau+2 (where tau is the least degree t such that the points" << endl;
<< "impose independent conditions in all degrees t or bigger)." << endl;
<< "Column 2 gives the value dim I_t of the Hilbert function" << endl;
<< "in each degree t listed in column 1. The resolution of I" << endl;
<< "is of the form 0 -> F_1 -> F_0 -> I -> 0, where F_1 and" << endl;
<< "F_0 are free S=k[P2] modules. Thus F_0=oplus_t S[-t]^{n_t}" << endl;
<< "and F_1=oplus_t S[-t]^{s_t} for integers s_t and n_t." << endl;
<< "Columns 3 and 4 give the values of n_t and s_t in each degree t" << endl;
<< "listed in column 1 (n_t and s_t are 0 in all other degrees)." << endl;
<< endl;

transpose(matrix({v0,v1,v3,v4})))

-- findhilb: computes e(F_t(Z)) for alpha-1<= t <=tau+1, which gives
-- a lower bound for the SHGH conjectural hilbert function
-- for a fat points subscheme involving general points of P2.
-- Call it as findhilb({m_1,...,m_n}) for integers m_i
-- specifying the multiplicities of the fat points in Z.
-- The conjecture is known to be correct for n<=9.

findhilb = (l) -> (
  ww=l;
  if #l<3 then ww=join(l,{0,0,0,0});
  n=#ww;
  n=n-1;
  ww=zw(ww);
  a1:=findalpha(ww);
  tau:=findtau(ww);
  d1:=a1-1;
  << "The output gives dim I_t, computed in degrees t from alpha(I)-1 to " << endl;
  << "reg(I), where tau = reg(I)-1 is least degree such that" << endl;
  << "hilbert function of I equals hilbert polynomial of I." << endl;
  if n>9 then (
    << "When more than 9 multiplicities are input," << endl;
    << "the output is a lower bound for dim I_t, which by the" << endl;
    << "SHGH conjecture should equal dim I_t." << endl);
  << endl;
  << " t " << " dim I_t " << " (tau = " << tau << ")" << endl;
  while (d1 <= tau+1) do (
    << " " << d1 << " " << homcompdim(fundom({d1,ww})) << endl;
    d1=d1+1)

--input: l=(n,m), n = number of points, m = uniform multiplicity
--output: various bounds on alpha and tau

unifbounds = (l) -> (
  n=l#0;
  m=l#1;
  ba=bestrda(n);
```

bb:=beststrdb(n);
ea:=uniffindalpha(l);
t:=0;
<< "number of general points n of P2: " << n << endl;
<< "multiplicity m of each point: " << m << endl;
<< endl;
if n<= 9 then (<< "Value of alpha: " << ea << endl)
else ( <<< "Expected value of alpha (via SHGH conjecture): " << ea << endl;
    " Note: The SHGH conjectural value of alpha is an upper bound." << endl);
<< "Lower Bounds on alpha:" << endl;
<< " Roe's, via unloading: " << unifroealpha(l) << endl;
tmp:=unifezbhalpha(l);
<< " Harbourne's, via Cor IV.i.2(a, b), using r="tmp#1" and d=";
<<tmp#2": " << tmp#0 << endl;
t=ea;
while(t==unifbhalpha(l,ba#0,ba#1,t)) do t=t-1;
<< " Harbourne's, via unloading, using r="<<ba#0" and d=";
<<ba#1": " << t+1 << endl;
d:=0;
while(d*d < n) do d=d+1;
d=d-1;
while(r**r < n*d*d) do r=r+1;
<< " Harbourne/Roe's first formula, using r="<<r<<" and d="<<
d": " << ezunifHRalpha(l,r,d) << endl;
r=d*d;
while(r*r < n*d*d) do r=r+1;
<< " Harbourne/Roe's second formula, using r="<<r<<" and d="<<
d": " << ezunifHRalphaB(l,r,d) << endl;
r=bb#0;
d=bb#1;
t=ea;
while(t==unifHRalpha(l,r,d,t)) do t=t-1;
t=t+1;
<< " Harbourne/Roe's (via modified unloading), using r="<<r<<" and d="<<d;
<<": " <<tt<<endl;
r=bb#0;
d=bb#1;
t=ea;
while(t==unifHRalpha(l,r,d,t)) do t=t-1;
t=t+1;
<< " Harbourne/Roe's (via modified unloading), using r="<<r<<" and d="<<d;
<<": " <<tt<<endl;
d=0;
while(d*d<n) do d=d+1;
d=d-2;
r=d*(d+1);
t=ea;
while(t==unifHRalpha(l,r,d,t)) do t=t-1;
t=t+1;
<< " Harbourne/Roe's (via modified unloading), using r="<<r<<" and d="<<d;
<<": " <<tt<<endl<<endl;
tt:=uniffindtau(l);
if n<= 9 then (<< "Value of tau: " << tt << endl)
else ( <<< "Expected value of tau (via SHGH conjecture): " << tt << endl;
    " Note: The SHGH conjectural value of tau is a lower bound." << endl);
<< "Upper Bounds on tau:" << endl;
<< " Hirschowitz's: " << Hiuniftau(l) << endl;
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<< " Gimigliano's: " << Guniftau(l) << endl;
if n>4 then ( << " Catalisano's: " << Cuniftau(l) << endl);
t=0;
while(t*(t+3)-n*m*(m+1) < 2*t*(m-1)-2) do t=t+1;
<< " Ballico's: " << t << endl;
t=0;
while(9*(t+3)*(t+3) <= 10*n*(m+1)*(m+1)) do t=t+1;
<< " Xu's: " << t << endl;
t=0;
tmp=0;
while(tmp*tmp < n) do tmp=tmp+1;
while(2*t < 2*m*tmp + tmp - 3) do t=t+1;
<< " Harbourne/Holy/Fitchett's: " << t << endl;
<< " Roe's, via unloading: " << unifroetau(l) << endl;
tmp=ezunifHRtau(l);
<< " Harbourne/Roe's first formula, using r="<<tmp#1<<" and d="<<
tmp#2<<": "<<tmp#0<<endl;
<< " Harbourne/Roe's second formula, using r="<<tmp#2*(tmp#2)<<" and d="<<
tmp#2<<": "<<ezunifHRtauB(l,tmp#2,tmp#2) << endl;
r=ba#0;
d=ba#1;
t=unifHRtau(l,r,d,tt);tmp=unifHRtau(l,bb#0,bb#1,tt);if tmp<t then (t=tmp;
r=bb#0;
d=bb#1);
<< " Harbourne/Roe's (via unloading), using r="<<r<<" and d="<<d<<": "<<t<<endl;
if (ba#0)*(bb#0)<(ba#1)*(bb#1)*n then (t = -3 + ceiling((m+1)*(bb#1)*n/(bb#0));
r=bb#0;
d=bb#1) else (t = -3 + ceiling((m+1)*(ba#0)/(ba#1));
r=ba#0;
d=ba#1);
<< " Via Ran's observation, and Harbourne's bound on alpha," << endl;
<< " using r="<<r<<" and d="<<d<<": "<<t<<endl)
-- input: l={m1,...,mn}, n >=1 (number of points), m1, ... >=1 (the multiplicities)
-- output: various bounds on alpha and tau

bounds = (l) -> (n:=#l;ba:=bestrda(n);bb:=bestrdb(n);ea:=findalpha(zr(l));
t:=0;tmp:=0;
<< "number of general points n of P2: " << n << endl;
<< "multiplicities of the points: " << l << endl;
<< endl;if n< 9 then ( << "Value of alpha: " << ea << endl)
else (<< "Expected value of alpha (via SHGH conjecture): " << ea << endl;
<< " Note: The SHGH conjectural value of alpha is an upper bound." << endl);
<< "Lower bounds on alpha:" << endl;
<< " Via Checking Psi: " << Psibound(l) << endl;
<< " Roe's, via unloading: " << roealpha(l) << endl;
w:=ezbhalphaA(l);
<< " Harbourne's, via Cor IV.i.2(a), using r="<<w#1<<", d=";
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<<\#2\#2\#2\#2\#2\#2\#2\#2\#2\#2<<: "<<\#0<<endl;
w=ezbhalphaB(l);

" Harbourne's, via Cor IV.i.2(b), using $r$="<<\#1<<", $d$="<<\#2<<": "<<\#0<<endl;
w=ezbhalphaD(l);

" Harbourne's, via Cor IV.i.2(d), using $r$="<<\#1<<", $d$="<<\#2<<": "<<\#0<<endl;
r:=ba#0;
d:=ba#1;
t=ea;
while(t==HRalpha(l,r,d,t)) do t=t-1;
t=t+1;
tmp=ea;
while(tmp==HRalpha(l,bb#0,bb#1,tmp)) do tmp=tmp-1;
tmp=tmp+1;
if tmp>t then ( t=tmp; r=bb#0; d=bb#1);
<< " Harbourne/Roe's (via modified unloading), using $r$="<<\#0<<": "<<\#1<<endl;
tt:=findtau(l);
if n= 9 then ( << "Value of tau: " << tt << endl)
else ( << "Expected value of tau (via SHGH conjecture): " << tt << endl;
<< " Note: The SHGH conjectural value of tau is a lower bound." << endl);
<< "Upper bounds on tau:" << endl;
<< " Hirschowitz's: " << Hitau(l) << endl;
<< " Gimigliano's: " << Gtau(l) << endl;

<< w=prmt(zr(l));
scan(#l, i->(if w#i >0 then nn=nn+1));
if nn>4 then ( << " Catalisano's: " << Ctau(l) << endl);
<< " Roe's, via unloading: " << roetau(l) << endl;
<< " Harbourne/Roe's (via unloading), using $r$="<<\#0<<": "<<\#1<<" and $d$="<<\#2<<": "<<\#0<<endl;
<< " One more bound on alpha; this one can be slow since it tries all r and d." << endl;
w=bestbhalpha(l,ea);
<< " Harbourne/Roe's (via unloading), using $r$="<<\#0<<": "<<\#1<<", $d$="<<\#2<<": "<<\#0<<endl)
--decomp: prints a decomposition $F=H+N$ for any divisor class $F$ in Psi
--as described in the print statements of the script.
--input: decomp(l), where $l=\{d,\{m_1,\ldots, m_n\}\}$, signifying the divisor
--class $F = dE_0-(m_1E_1+\cdots+m_nE_n)$.

decomp = (1) -> ( << "Let Psi be the subsemigroup of divisor classes generated by " << endl;
<< "exceptional classes and by $-K$. For any divisor class $F$, this " << endl;
<< "script determines if $F$ is in Psi, and if so gives a decomposition" << endl;
<< "F=H+N, where $N$ is a sum of exceptionals orthogonal to each other and to $H$" << endl;
<< "and $H$ is in Psi but $H.E \geq 0$ for all exceptionals $E$. The point of this" << endl;
is that dim |F| = dim |H|, and conjecturally dim |H| = (H^2-H.K)/2.

```
i:=0;
j:=0;
w:=1;
v:={};
ww:={};
ex:={};
mult:=0;
tmp:=fundom(l);
if tmp#0<tmp#1#0 or tmp#0<0 then (<< "Your class is not in \( \Psi \)." << endl) else (  
    if #l#1<3 then w={l#0,join(l#1,{0,0,0})};
    d:=3*(#(w#1)); --define an element \( \{d,v\} \) in fundamental domain
    scan(#(w#1), i->(v=join(v,{{#(w#1)} - i})));
    ww=fundomboth(w,{d,v});
    if ww#0#1 == zr(ww#0#1) then (<< "N = 0" << endl ) else (  
      << "N is a sum of the following fixed exceptional classes:" << endl;
      if (ww#0#0) - (ww#0#1#0) - (ww#0#1#1) < 0 then (  
        scan(#(w#1), j->(if j<=1 then ex=join(ex,{1}) else ex=join(ex,{0})));
        ex={1,ex};
        ex=(fundomboth(ww#1,ex))#1;
        <<ex" is a fixed component of multiplicity ";
        <<ww#0#1) - (ww#0#1#0) - (ww#0#1#1) << endl;
        scan(#(w#1), j->(if j>1 then ex=join(ex,ww#0#1#j)));
        ww={2*(ww#0#0)-(ww#0#1#0)-(ww#0#1#1),ex},ww#1});
    scan(#(w#1), i->(  
      if (ww#0#1)#i<0 then (  
        ex={};
        mult=-(ww#0#1)#i;
        scan(#(w#1), j->(if j==i then ex=join(ex,{-1}) else ex=join(ex,{0})));
        ex={0,ex};
        ex=(fundomboth(ww#1,ex))#1;
        << ex " is a fixed component of multiplicity " << mult << endl))));
    << endl << "and \( H = " << (fundomboth(ww#1,{ww#0#0,zr(ww#0#1)})#1 << endl))
```

-- input: list l of multiplicities for fat points Z  
-- output: least \( t \) such that \( F_t(Z) \) is in \( \Psi \)

```
Psibound = (l) -> (  
t:=findalpha(l);
  tmp:=fundom({t,l});
  while(tmp#0>=tmp#1#0 and tmp#0>=0) do (t=t-1;
     tmp=fundom({t,l}));
  t+1)
```

-- prmt: arranges the elements of the list \( l=\{m_1,...,m_n\} \) in descending order  
-- Call it as prmt(l) where l is a list of integers.

```
prmt = (l) -> (  
  (prmtboth(l,l))#0)
```

-- prmtboth: arranges the elements of the list l1 in descending order,  
-- and applies the same permutation to l2  
-- Call it as prmt(l1,l2) where l1 and l2 are lists of integers.

```
prmtboth = (l1,l2) -> (  
  tmpv1:l1;
  tmpv2:l2;
  v1:l1;
  v2:l2;
  ```
i:=0;
j:=0;
k:=0;
scan(#l1, i->(scan(#l1, j->(
if tmpv1#i < tmpv1#j then (
if i < j then (
    k=-1;
    v1={};
    v2={};
    while(k<#l1-1) do (k=k+1;
      if k==i then (v1=join(v1,{tmpv1#j});
        v2=join(v2,{tmpv2#j})) else (v1=join(v1,{tmpv1#k});
        v2=join(v2,{tmpv2#k}));
    tmpv1=v1;
    tmpv2=v2)));

  {v1,v2})))

-- zr: replaces negative values in a list l by zeroes.
-- Call it as zr(l) where l is a list of integers.

zr = (l) -> (
v:={};
i:=0;
scan(#l, i->(if l#i<0 then v=join(v,{0}) else v=join(v,{l#i}))); v)

-- quad: performs a quadratic transform on a divisor class dE_0-(m_1E_1+...+m_nE_n).
-- Call it as quad({d,{m1,...,mn}}). The output is
-- {2d-m1-m2-m3,m2-d3-d4+m1,m2-d3-d4+m1,m3-d2-d4+m1,m3-d2-d4+m1,..,mn}.

quad = (l) -> (
  i:=0;
w:=l;
  if #l#1<3 then w={l#0,join(l#1,{0,0,0})};

  v:={w#0 - w#1#1 - w#1#2,w#0 - w#1#0 - w#1#2,w#0 - w#1#0 - w#1#1};

  scan(#w#1, i->(if i>2 then v=join(v,{w#1#i}))); v
)

-- fundom: Call it as fundom({d,{m1,...,mn}}). The output is a new
-- list {d',{m1',...,mn'}}; the class dE_0-(m_1E_1+...+m_nE_n) is
-- equivalent via Cremona transformations to d'E_0-(m_1'E_1+...+m_n'E_n),
-- where d' is either negative or as small as possible.

fundom = (l) -> ((fundomboth(l,l))#0)

-- fundomboth: applies fundom to l1 to reduce l1 to fundamental
-- domain of a certain group operation, and applies the same
-- group operation g to l2. If l2 starts out in the fundamental domain,
-- and {l1',l2'}=fundomboth(l1,l2), then {l2,g^{-1}l}=fundomboth(l2',l).
-- This allows one to compute the action of g^{-1}.

fundomboth = (l1,l2) -> (w1:=l1;
w2:=l2;
v:={});
if #11#1<3 then w1={l1#0,join(l1#1,{0,0,0})};
if #12#1<3 then w2={l2#0,join(l2#1,{0,0,0})};
v=prmtboth(w1#1,w2#1);
w1={w1#0,v#0};
w2={w2#0,v#1};
while ((w1#0 < w1#1#0 + w1#1#1 + w1#1#2) and (w1#0 >= 0)) do (w1=quad(w1);
w2=quad(w2);
v=prmtboth(w1#1,w2#1);
w1={w1#0,v#0};
w2={w2#0,v#1});
{w1,w2})

-- homcompdim: computes e(F_t(Z)), the expected dimension of a component I_d
-- of a fat points ideal I corresponding to a fat point subscheme Z of general
-- points taken with multiplicities m_1, ..., m_n. Call it as
-- homcompdim({d,{m_1,...,m_n}}); the output is the SHGH conjectural
-- dimension of I_d, which is the actual dimension if n < 10.

homcompdim = (l) -> (h:=0; i:=0; w:=l;
if #l#1<3 then w={l#0,join(l#1,{0,0,0})};
w=fundom(w);
d:=w#0;
w=fundom({d,zr(w#1)});
d=w#0;
v:=zr(w#1);
if d<0 then h=0 else (tmp:=0;
scan(#v, i->(tmp = v#i*v#i+v#i+tmp));
h=floor((d*d+3*d+2-tmp)/2);
if h < 0 then h=0);
h)

-- findalpha: find alpha, the least degree t such that
-- I_t \ne 0, where I is the ideal corresponding to n general
-- points taken with multiplicities m_1, ..., m_n. Call it as
-- findalpha({m_1,...,m_n}). The output is the SHGH conjectural value
-- of alpha, which is the actual value if n < 10 and an upper bound otherwise.

findalpha = (l) -> (i:=1;
w:=prmt(zr(1));
if #l<3 then w=join(l,{0,0,0});
d:=w#0; -- alpha is at least the max mult
if (#w)<9 then ( -- if n<8, to speed things, make an estimate
while(i < (#w)) do (d=d+w#i;
  i=i+1);
  d=ceiling(d/3);
if d < w#0 then d = w#0);
while (homcompdim({d,w}) < 1) do d=d+1;
d)

-- dot: computes a dot product of two lists l1 and l2 (of equal length)
-- of integers. Call it as dot(l1,l2).
dot = (l1,l2) -> (i:=0;
dottot := 0;
scan(#l1, i->(dottot = dottot + (l1#i)*(l2#i)));
dottot

-- input: l={m1,...,mn}, n \geq 1 (number of points), m1,... (the multiplicities)
-- output: the SHGH conjectured value of tau; this is the actual value if
-- n < 10, and a lower bound otherwise.

findtau = (l) -> (
  t:=findalpha(l);
  if t > 0 then t = t-1; -- tau is at least alpha - 1
  n:=#l;
  v:=l;
  p:=dot(v,v);
  K:={};
  j:=0;
  q:=0;
  scan(#l, j->(q=q+l#j));
  while(2*(homcompdim(join({t},{v}))) > t*t-p+3*t-q+2) do t=t+1;
  t)

-- input: positive integer n
-- output: {r,d}, where r^2 \geq d^2n, n \geq r, and nd/r is as big as possible

bestrda = (n) -> (n)
  rootn:=0;
  while(rootn*rootn< n) do rootn=rootn+1;
  rootn=rootn-1;
  d:=1;
  r:=0;
  if rootn*rootn==n then r=rootn else r=rootn+1;
  tmpr:=1;
  tmpd:=1;
  while(tmpd<=rootn) do (tmpr=tmpd*rootn;
    while(tmpr*tmpr<tmpd*tmpd*n) do tmpr=tmpr+1;
    if tmpr*d < tmpd*r then (r=tmpr;
      d=tmpd);
    tmpd=tmpd+1);
{l}

-- input: positive integer n
-- output: {r,d}, where r^2 \leq d^2n, n \geq r, and r/d is as big as possible

bestrdb = (n) -> (n)
  rootn:=0;
  while(rootn*rootn< n) do rootn=rootn+1;
  d:=1;
  r:=0;
  if rootn*rootn==n then r=rootn else r=rootn-1;
  tmpr:=1;
  tmpd:=1;
  while(tmpd<rootn) do (tmpr=tmpd*rootn;
    while(tmpr*tmpr>tmpd*tmpd*n) do tmpr=tmpr-1;
    if tmpr>n then tmpr=n;
    if tmpr*d > tmpd*r then (r=tmpr;
      d=tmpd);
    tmpd=tmpd+1);
roaelpha = (l) -> (  
  i:=0;  
  v={};  
  w2={};  
  w:=l;  
  i1:=2;  
  if #l<3 then w=join(w,{0,0,0});  
  while(i1<#w) do (  
    v={1};  
    scan(#w, i->(if i>0 then (if i<=i1 then v=join(v,{-1}) else v=join(v,{0})));  
    w=zr(prmt(w));  
    while(dot(w,v)<0) do (  
      w2={};  
      scan(#w, i->(w2 = join(w2,{w#i+v#i})));  
      w=zr(prmt(w2));  
      i1=i1+1;  
    end;  
    w#0)  
  end;  
)

unifroaelpha = (l) -> (  
  i1:=0;  
  intchk:=0;  
  n:=l#0;  
  m:=l#1;  
  roebnd:=m;  
  q:=n-1;  
  if n>2 then (  
    i1=2;  
    while(i1<n) do (  
      if q<i1 then intchk=i1*m-(i1-q) else intchk=i1*m;  
      while(roebnd<intchk) do (  
        roebnd=roebnd+1;  
        if q<i1 then (q=n-i1+q-1;  
          m=m-1) else (q=q-i1;  
          if q==0 then (m=m-1;  
            q=n-1));  
        if q<i1 then intchk=i1*m-(i1-q) else intchk=i1*m);  
      end;  
      i1=i1+1)  
    end;  
    roebnd  
  end;  
)

roetau = (l) -> (  
  i:=0;  
  vv={};  
  vv1:={};  
  w2={};  
  ww:=l;  
  i1:=1;  
  while(i1<#ww-1) do (  

vv={1};
scan(#ww, i->(if i>0 then (if i=1 then vv=join(vv,{-1}) else vv=join(vv,{0}))));
vv1={1};
scan(#ww, i->(if i>0 then (if i=1 then vv1=join(vv1,{-1}) else vv1=join(vv1,{0}))));
ww=zar(prmt(ww));
while((dot(ww,vv)) < -1) do ( 
   w2={};
   scan(#ww, i->(w2 = join(w2,{ww#i+vv1#i}))); 
   ww=zar(prmt(w2)); 
   i1=i1+1;
ww#0+ww#1-1)
-- input: l={n,m}, n >=1 (number of points), m >=1 (uniform multiplicity)
-- output: Roe’s algorithmic upper bound on tau

unifroetau = (l) -> ( 
   i:=1; 
   s:=0; 
   n:=l#0; 
   m1:=l#1; 
   m2:=0; 
   if n>1 then m2=m1; 
   n2:=n-1; -- n2 keeps track during unloading of number of 
      -- points with multiplicity m2 
   while(i < (n-1)) do ( 
      if (i+1) <= n2 then s=(i+1)*m2 else s=(i+1)*m2+n2-i-1; 
      while((m1-s) < -1) do ( 
         m1=m1+1; 
         if i < n2 then n2=n2-i else ( 
            n2=n-i+n2-1; 
            m2=m2-1); 
         if (i+1) <= n2 then s=(i+1)*m2 else s=(i+1)*m2+n2-i-1; 
         i=i+1; 
      m1+m2-1)
   -- input: l={n,m}, n >=1 (number of points), m >=1 (uniform multiplicity)
   -- output: list {i,r,d}, with i being Harbourne's easy lower bound on 
      -- alpha (via Cor IV.i.2 (a), (b)) computed using the best r and d 

unifezhhalpha = (l) -> ( 
   w:=bestrda(l#0); 
   i:=ceiling((l#1)*(l#0)*(w#1)/(w#0))); -- compute mnd/r rounded up 
   r:=w#0; 
   d:=w#1; 
   w:=bestrdb(l#0); 
   j:=ceiling((l#1)*(w#0)/(w#1))); -- compute mr/d rounded up 
   if i<j then (i=j; 
      r=w#0; 
      d=w#1); 
   {i,r,d})

-- input: ((n,m),r,d), n >=1 (number of points), m >=1 (uniform multiplicity)
-- where r <= n, and 2r >= n+d^2 
-- output: lower bound on alpha via formula of [HR]

ezunifHRalpha = (l,r,d) -> ( 
   n:=l#0; 
   m:=l#1; 
   t:=0; 
   q:=ceiling(n*m/r)-1;
while(((t+2)*(t+1)<=2*(m*n-r*q)) and t<d) do t=t+1; 
t+q*d)

-- input: ({n,m},r,d), n >=1 (number of points), m >=1 (uniform multiplicity)
-- where r <= n, d(d+1)/2 <= r, and r^2 <= nd^2
-- output: lower bound on alpha via formula of [HR2].

ezunifHRalphaB = (l,r,d) -> ( 
n:=l#0; 
m:=l#1; 
g:=(d-1)*(d-2)/2; 
tmp:=floor((m*r+g-1)/d); 
t:=ceiling(n*m/r)-1; 
rr:=m*n-r*t; 
s:=0; 
while(((s+1)*(s+2) <= 2*rr) and s<d) do s=s+1; 
s=s-1; 
t=s+t*d; 
if tmp<t then t=tmp; 
t+1)

-- input: ({n,m},r,d), n >=1 (number of points), m >=1 (uniform multiplicity)
-- where r <= n, and r <= d^2
-- output: upper bound on tau via formula of [HR].

ezunifHRtauB = (l,r,d) -> ( 
n:=l#0; 
m:=l#1; 
g:=(d-1)*(d-2)/2; 
q:=ceiling(n*m/r)-1; 
rr:=m*n-r*q; 
t:=q*d+ceiling((rr+g-1)/d); 
tmp:=q*d+d-2; 
if t<tmp then t=tmp; 
t)

-- input: ({n,m},r,d,ea), n >=1 (number of points), m >=1 (uniform multiplicity)
-- ea is an estimate for alpha (for speed); it must be set equal to
-- a value no bigger than the eventual value of unifHRalpha (0, for example)
-- output: lower bound on alpha via modified unloading method of [HR],
-- computed using given r and d

unifHRalpha = (l,r,d,ea) -> ( 
i:=ea-1; 
n2:=0; -- n2 keeps track of the number of points with maximum multiplicity 
q:=ceiling(n*m/r)-1; 
rr:=m*n-r*q; 
t:=q*d+ceiling((rr+g-1)/d); 
tmp:=q*d+d-2; 
if t<tmp then t=tmp; 
t)

-- input: ({n,m},r,d,ea), n >=1 (number of points), m >=1 (uniform multiplicity)
-- ea is an estimate for alpha (for speed); it must be set equal to
-- a value no bigger than the eventual value of unifHRalpha (0, for example)
-- output: lower bound on alpha via modified unloading method of [HR],
-- computed using given r and d

unifHRalpha = (l,r,d,ea) -> ( 
i:=ea-1; 
n2:=0; -- n2 keeps track of the number of points with maximum multiplicity 
q:=ceiling(n*m/r)-1; 
rr:=m*n-r*q; 
t:=q*d+ceiling((rr+g-1)/d); 
tmp:=q*d+d-2; 
if t<tmp then t=tmp; 
t)
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tmpm=tmpm-1;
if tmpm <= 0 then (tmpm=0;
n2=l#0));
if r<=n2 then s=r*tmpm else s=r*tmpm-r+n2));

-- input: ($m_1, \ldots, m_n, r, d, ea$), $n \geq 1$ (number of points), $m_i \geq 1$ (multiplicities)
-- ea is an estimate for alpha (for speed); it must be set equal to
-- a value no bigger than the eventual value of HRalpha (0, for example)
-- output: lower bound on alpha via modified unloading method of [HR],
-- computed using given r and d

HRalpha = (l,r,d,ea) -> (i:=ea-1;
j:=0;
n:=0;
g:=(d-1)*(d-2)/2;
v:=prmt(zr(l));
tmpv:=v;
ttmpv:={};
scan(#l, j->(if v#j>0 then n=n+1));
if n>0 then (
ww:={};
scan(#l, j->(if j<r then ww=join(ww,{1}) else ww=join(ww,{0})));
tmpi:=i-1;
while(tmpi < tmpv#0) do (i=i+1;
tmpi=i;
tmpv=v;
while(((tmpi*d-(dot(ww,tmpv))<g) and (tmpi >= d-2)) or
((tmpi+1)*(tmpi+2)<=2*(dot(ww,tmpv))) and (tmpi<d) and (tmpi>=0))) do (tmpi=tmpi-d;
ttmpv={};
scan(#l, j->(ttmpv=join(ttmpv,((tmpv#j)-(ww#j))));
tmpv=prmt(zr(ttmpv))));

-- input: ($n, m, r, d, ea$), $n \geq 1$ (number of points), $m \geq 1$ (uniform multiplicity)
-- ea is an estimate for alpha (for speed); it must be set equal to
-- a value no bigger than the eventual value of unifbalpha (0, for example)
-- output: Harbourne's algorithmic lower bound on alpha via unloading,
-- using the given r and d.

unifbalpha = (l,r,d,ea) -> (i:=ea-1;
n2:=0; -- n2 keeps track of the number of points with maximum multiplicity
s:=0;
tmpi:=-1;
tmpm:=0;
while(tmpi<0) do (i=i+1;
tmpi=i;
tmpm=l#1;
n2=l#0;
if r <= n2 then s=r*tmpm else s=r*tmpm-r+n2;
while((tmpi+d-s < 0) and (tmpi >= 0)) do (tmpi=tmpi-d;
if r<n2 then n2=n2-r else (n2=l#0-r+n2;
tmpm=tmpm-1;
    if tmpm <= 0 then (  
      tmpm=0;
      n2=1#0);
    if r<=n2 then s=r*tmpm else s=r*tmpm-r+n2));

-- input: ({n,m},r,d), n >=1 (number of points), m >=1 (uniform multiplicity)
-- where r <= n, and 2r >= n+d^2
-- output: lower bound on alpha via formula which agrees with that via unloading.

ezunifBHalphaB = (l,r,d) -> (  
  n:=l#0;
  m:=l#1;
  q:=floor(n*m/r);
  rr:=m*n-r*q;
  t:=q-1+ceiling(rr/d);
  tmp:=d*ceiling(n*m/r);
  if tmp<t then t=tmp;
  t+1)

-- input: l={m1,...,mn}, n >=1 (number of points), m1, ... >=1 (the multiplicities)
-- output: list (aa,r,d), with aa being Harbourne's lower bound on
-- alpha (via Cor IV.i.2(a)), computed using the best r and d

ezbhalphaA = (l) -> (  
  i:=0;
  s:=0;
  n:=0;
  v:=prmt(zr(l));
  scan(#l, i->(if v#i>0 then n=n+1));
  w:=bestrda(n);
  i=0;
  while(i<n) do (  
    s=s+v#i;
    i=i+1);
  best:=ceiling(s*(w#1)/(w#0));
  {best, w#0, w#1})

-- input: l={m1,...,mn}, n >=1 (number of points), m1, ... >=1 (the multiplicities)
-- output: list (aa,r,d), with aa being Harbourne's lower bound on
-- alpha (via Cor IV.i.2(b)), computed using the best r and d

ezbhalphaB = (l) -> (  
  i:=0;
  s:=0;
  n:=0;
  v:=prmt(zr(l));
  scan(#l, i->(if v#i>0 then n=n+1));
  w:=bestrdb(n);
  i=0;
  while(i<n) do (  
    s=s+v#i;
    i=i+1);
  best:=ceiling(s*(w#0)/(n*(w#1)));  
  {best, w#0, w#1})

-- input: l={m1,...,mn}, n >=1 (number of points), m1, ... >=1 (the multiplicities)
-- output: list (aa,r,d,j), with aa being Harbourne's lower bound on
-- alpha (via Cor IV.i.2(d)), computed using the best r, d and j
ezbhalphad = (l) -> {
  w:=\{
  i:=0;
  scan(#l, i->(if l#i>0 then w=join(w,{l#i})));
  w=prmt(w);
  n:=#w;
  bnd:=0;
  tmpbnd:=0;
  r:=0;
  d:=0;
  j:=0;
  tmpr:=0;
  tmpd:=0;
  tmpj:=0;
  if n>0 then {
    while(tmpr<n) do (tmpr=tmpr+1;
      tmpd=0;
      while(tmpd*tmpd<tmpr) do (tmpd=tmpd+1;
        tmpj=0;
        while(tmpj<tmpd*tmpd) do (tmpj=tmpj+1;
          tmpbnd=lpa(w,tmpr,tmpd,tmpj);
          if tmpbnd>bnd then (
            bnd=tmpbnd;
            r=tmpr;
            d=tmpd;
            j=tmpj))));
  }{bnd,r,d,j})

-- lpa computes the bound given in Cor IV.i.2(d); attempts
-- various solutions with the hope of approximating the optimal
-- solution to the linear programming problem indicated by Thm IV.i.1.
-- lpa is called by ezbhalphad

lpa = (l,r,d,j) -> {
  i:=0;
  n:=#l;
  sum:=0;
  sumb:=0;
  bnd:=0;
  if d*d >= r then {
    scan(#l, i->(if i<r then sum=sum+l#i));
    bnd=ceiling(sum/d)) else {
      if j==0 then {
        scan(#l, i->(if i<d*d then sum=sum+l#i));
        bnd=ceiling(sum/d)) else {
          M:=floor((r-d*d)*(r-d*d+j)/j);
          scan(#l, i->(if i<d*d-j then sum=sum+l#i else (if i<M+r then sumb=sumb+l#i)));
          sumb=sumb*j/(r-d*d+j);
          if M<n-r then sumb=sumb+(l#(M+r))*(r-d*d-j*M/(r-d*d+j));
          bnd=ceiling((sumb+sum)/d)));
    }bnd
  }

-- input: ((m1,...,mn),r,d,ea), n >=1 (number of points), m1, ... >=1
-- (the multiplicities), r and d positive integers, ea any value
-- not bigger than the eventual value of bhalpha; can be set to 0
-- output: Harbourne's unloading lower bound on alpha, using given r and d

bhalpha = (l,r,d,ea) -> {
  i:=ea-1;
j:=0;
n:=0;
v:=prmt(zr(l));
tmpv:=v;
ttmpv:={};
scan(#l, j->(if v#j>0 then n=n+1));
if n>0 then (  
  ww:={};
  scan(#l, j->(if j<r then ww=join(ww, {1}) else ww=join(ww, {0})));
  tmpr:=0;
  tmpd:=0;
  if n>0 then (    
    while(tmpr<n) do (tmpr=tmpr+1;
      tmprd:=0;
      while(tmprd<tmpr) do (tmprd=tmprd+1;
        tmpbnd:=ea;
        while(tmpbnd=bhalpha(w,tmpr,tmpd,tmpbnd)) do tmpbnd=tmpbnd-1;
        tmpbnd=tmpbnd+1;
        if tmpbnd>bnd then (  
          bnd=tmpbnd;
          r=tmpr;
          d=tmpd));

{bnd,r,d})

-- Find bhalpha using best possible r and d;
-- ea is an a priori estimate for alpha (for speed)
-- it must be set to a value >= than the actual value
-- of alpha (e.g., ea=findalpha(l))

bestbhalpha = (l,ea) -> (  
  w:={};
  i:=0;
  scan(#l, i->(if l#i>0 then w=join(w, {l#i})));  
  w=prmt(w);
  n:=#w;
  bnd:=0;
  tmpr:=0;
  tmpd:=0;
  if n>0 then (    
    while(tmpr<n) do (tmpr=tmpr+1;
      tmprd=0;
      while(tmprd<tmpr) do (tmprd=tmprd+1;
        tmpbnd:=ea;
        while(tmpbnd=bhalpha(w,tmpr,tmpd,tmpbnd)) do tmpbnd=tmpbnd-1;
        tmpbnd=tmpbnd+1;
        if tmpbnd>bnd then (  
          bnd=tmpbnd;
          r=tmpr;
          d=tmpd));

{bnd,r,d})

-- input: (m1, ..., mn), r, d, et, n >=1 (number of points), m1, ... >=1 (the multiplicities),
-- r and d positive integers, et any lower bound for tau (used for speed; can be set to 0).
-- output: Harbourne/Roe's algorithmic upper bound on tau,
-- with given r and d (assumes char = 0).

HRtau = (l,r,d,et) -> (  
  i:=et-1;
  j:=0;
  n:=0;

-- input: (m1, ..., mn), r, d, et, n >=1 (number of points), m1, ... >=1 (the multiplicities),
-- r and d positive integers, et any lower bound for tau (used for speed; can be set to 0).
-- output: Harbourne/Roe's algorithmic upper bound on tau,
-- with given r and d (assumes char = 0).
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\[ v := \text{prmt}(zr(1)); \]
\[ \text{tmpv} := v; \]
\[ \text{ttmpv} := \emptyset; \]
\[ \text{scan}(#l, j \rightarrow \text{if } v#j > 0 \text{ then } n = n + 1); \]
\[ \text{if } n > 0 \text{ then (} \]
\[ \text{ww} := \emptyset; \]
\[ g := (d-1)*(d-2)/2; \quad \text{-- genus of plane curve of degree } d \]
\[ \text{scan}(#l, j \rightarrow \text{if } j < r \text{ then } \text{ww} = \text{join}(\text{ww}, \{1\}) \text{ else } \text{ww} = \text{join}(\text{ww}, \{0\})); \]
\[ \text{tmpi} := 0; \]
\[ \text{while} (\text{ttmpv}#0 > 0) \text{ do (} \]
\[ i = i + 1; \]
\[ \text{tmpi} = i; \]
\[ \text{tmpv} = v; \]
\[ \text{while} ((\text{tmpi} \cdot d - \text{dot}(\text{ww}, \text{tmpv})) \geq g - 1) \text{ and } (\text{tmpi} \geq d - 2) \text{ and } (\text{tmpv}#0 \geq 0) \text{ do (} \]
\[ \text{tmpi} = \text{tmpi} - d; \]
\[ \text{ttmpv} := \emptyset; \]
\[ \text{scan}(#l, j \rightarrow (\text{ttmpv} = \text{join}(\text{ttmpv}, ((\text{tmpv}#j) - (\text{ww}#j)))); \]
\[ \text{tmpv} = \text{prmt}(zr(\text{ttmpv}))); \]
\[ i) \]

-- input: $({n, m}, r, d, \text{et})$, $n \geq 1$ (number of points), $m \geq 1$ (the uniform multiplicity),
-- $r$ and $d$ positive integers, $\text{et}$ any lower bound for $\tau$ (used for speed; can be set to 0)
-- output: Harbourne/Roe’s algorithmic upper bound on $\tau$, using given $r$ and $d$
-- (assumes char = 0).

\[ \text{unifHRtau} = (l, r, d, \text{et}) \rightarrow ( \]
\[ i := \text{et} - 1; \]
\[ n2 := 0; \quad \text{-- } n2 \text{ is the number of points with maximum multiplicity} \]
\[ s := 0; \]
\[ \text{tmpm} := 1; \]
\[ \text{tmpi} := 0; \]
\[ g := (d-1)*(d-2)/2; \quad \text{-- genus of plane curve of degree } d \]
\[ \text{while} (\text{tmpm} > 0) \text{ do (} \]
\[ i = i + 1; \]
\[ \text{tmpi} = i; \]
\[ \text{tmpm} = l#1; \]
\[ n2 = 1#0; \]
\[ \text{if } r = n2 \text{ then } s = r \cdot \text{tmpm} \text{ else } s = r \cdot \text{tmpm} - r \cdot n2; \]
\[ \text{while} ((\text{tmpi} \cdot d - s) \geq g - 1) \text{ and } (\text{tmpi} \geq d - 2) \text{ and } (\text{tmpm} > 0) \text{ do (} \]
\[ \text{tmpi} = \text{tmpi} - d; \]
\[ \text{if } r < n2 \text{ then } n2 = n2 - r \text{ else (} \]
\[ n2 = 1#0 - r \cdot n2; \]
\[ \text{tmpm} = \text{tmpm} - 1; \]
\[ \text{if } \text{tmpm} < 0 \text{ then (} \]
\[ \text{tmpm} = 0; \]
\[ n2 = 1#0); \]
\[ \text{if } r <= n2 \text{ then } s = r \cdot \text{tmpm} \text{ else } s = r \cdot \text{tmpm} - r \cdot n2); \]
\[ i) \]

-- input: $l = \{n, m\}$, $n \geq 1$ (number of points), $m \geq 1$ (the uniform multiplicity)
-- output: list $\{a, r, d\}$, where $a$ is Harbourne/Roe’s formulaic upper
-- bound on $\tau$ (char 0) computed using $r$ and $d$

\[ \text{ezunifHRtau} = (l) \rightarrow ( \]
\[ n := l#0; \]
\[ m := l#1; \]
\[ d := 0; \]
\[ \text{while } (d \cdot d <= n) \text{ do } d = d + 1; \]
\[ d = d - 1; \]
\[ r := d; \]
while (r*r < d*d*n) do r=r+1;
g:=(d-2)*(d-1)/2;
a:=-ceiling((m*r+g-1)/d);
b:=-2*d*ceiling(m*n/r);
if a<b then a=b;
{a,r,d})

-- input: l={n,m}, n >=1 (number of points), m >=1 (the uniform multiplicity)
-- output: the SHGH conjectured value of alpha; this is the actual value if
-- n < 10, and an upper bound otherwise.

uniffindalpha = (l) -> (  
n:=l#0;
m:=l#1;
a:=-1;
if n=1 then a=m;
if n=2 then a=m;
if n=3 then a=ceiling(3*m/2);
if n=4 then a=2*m;
if n=5 then a=2*m;
if n=6 then a=ceiling(12*m/5);
if n=7 then a=ceiling(21*m/8);
if n=8 then a=ceiling(48*m/17);
if n=9 then a=3*m;
if n>9 then (  
    while(a*a-n*m*m+3*a-n*m+2 <0) do a=a+m;
    a=a-m;
    while(a*a-n*m*m+3*a-n*m+2 <=0) do a=a+1);
a)

-- input: l={n,m}, n >=1 (number of points), m >=1 (the uniform multiplicity)
-- output: the SHGH conjectured value of tau; this is the actual value if
-- n < 10, and a lower bound otherwise.

uniffindtau = (l) -> (  
n:=l#0;
m:=l#1;
t:=-1;
if n=1 then t=m-1;
if n=2 then t=2*m-1;
if n=3 then t=2*m-1;
if n=4 then t=2*m;
if n=5 then t=ceiling((5*m-1)/2);
if n=6 then t=ceiling((5*m-1)/2);
if n=7 then t=ceiling((8*m-1)/3);
if n=8 then t=ceiling((17*m-1)/6);
if n=9 then t=3*m;
if n>9 then (  
    while(t*t-n*m*m+3*t-n*m+2 <0) do t=t+m;
    t=t-m;
    while(t*t-n*m*m+3*t-n*m+2 <=0) do t=t+1);
    if t<0 then t=0;
t)

-- input: l={n,m}, n >=1 (number of points), m (the multiplicity of each point)
-- output: Hirschowitz's lower bound for tau

Hiuniftau = (l) -> (  
n:=l#0;
m:=l#1;
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```
t:=m;
s:=n*m*(m+1);
a:=ceiling((t+3)/2);
b:=ceiling((t+2)/2);
while(a*b*2 <= s) do (  
t=t+1;
    a=ceiling((t+3)/2);
    b=ceiling((t+2)/2));
t)
```

-- input: $l=${m1,...,mn}, n $\geq$1 (number of points), m1, ... $\geq$1 (the multiplicities)
-- output: Hirschowitz's lower bound for $\tau$

```
Hitau = (l) -> (  
n:=#l;
i:=0;
w:=prmt(zr(l));
t:=w#0;
s:=0;
scan(#l, i->(s=s+(w#i)*((w#i)+1)));
a:=ceiling((t+3)/2);
b:=ceiling((t+2)/2);
while(a*b*2 <= s) do (  
t=t+1;
    a=ceiling((t+3)/2);
    b=ceiling((t+2)/2));
t)
```

-- input: $l=${n,m}, n $\geq$1 (number of points), m (the multiplicity of each point)
-- output: Gimigliano's lower bound for $\tau$

```
Guniftau = (l) -> (  
n:=l#0;
m:=l#1;
t:=0;
while(t*(t+3)<2*n) do t=t+1;
m*t)
```

-- input: $l=${m1,...,mn}, n $\geq$1 (number of points), m1, ... $\geq$1 (the multiplicities)
-- output: Gimigliano's lower bound for $\tau$

```
Gtau = (l) -> (  
n:=0;
w:=prmt(zr(l));
scan(#l, i->(if w#i >0 then n=n+1));
t:=0;
s:=0;
i:=0;
while(t*(t+3)<2*n) do t=t+1;
scan(#l, i->(if i<t then s=s+w#i));
s)
```

-- input: $l=${n,m}, n $\geq$5 (number of points), m $>$0 (the multiplicity of each point)
-- output: Catalisano's lower bound for $\tau$

```
Cuniftau = (l) -> (  
s:=l#0;
m:=l#1;
r:=0;
t:=0;
```

```
f:=0;
while(f*(f+1) <= 2*s) do f=f+1;
f:=f-1;
while(2*r<2*s-f*(f+1)) do r=r+1;
d1:=0;
d:=f;
if r==0 then d1=f-1 else d1=f;
t:=d1+(m-1)*d;
if 2*t+1 < 5*m then t:=ceil((5*m-1)/2);
if t<2*m-1 then t:=2*m-1;
if r == f then (if s >= 9 then t:=m*d1+1);
t)

-- input: l={m1,...,mn}, n >=5 (number of points), m1, ... >=1 (the multiplicities)
-- output: Catalisano's lower bound for tau

Ctau = (l) -> (n:=0;
i:=0;
w:=prmt(zr(l));
scan(#l, i->(if w#i >0 then n=n+1));
vm:={};
vs:={};
i=0;
while(i < n-1) do (
  if w#i > w#(i+1) then (vs=join(vs,{i+1});vm=join(vm,{w#i});i=i+1);
  vs=join(vs,{n});vm=join(vm,{w#(n-1)});
i=#vm - 1;
v:={vm#i};
while(i > 0) do (
  v=join({vm#(i-1) - vm#i},v);
i=i-1);
vf:={};
vr:={};
scan(#vs, i->(f:=0;
r:=0;
while(f*(f+1) <= 2*(vs#i)) do f=f+1;
f:=f-1;
while(2*r<2*(vs#i)-f*(f+1)) do r=r+1;
vf=join(vf,{f});
vr=join(vr,{r}));
t:=0;
if (vr#(#vr-1)) == 0 then t = - 1;
d1:=t+vf#(#vf-1);
scan(#vs, i->(t=t+(vf#i)*(v#i)));
if 2*t+1 < (w#0)+(w#1)+(w#2)+(w#3)+(w#4) then
t:=ceil(((w#0)+(w#1)+(w#2)+(w#3)+(w#4)-1)/2);
if t<(w#0)+(w#1)-1 then t=(w#0)+(w#1)-1;
if (vr#0) == (vf#0) then (if s >= 9 then (if (w#0)==(w#(n-1))
  then (if (w#(n-1)) == 1 then t=(w#0)*d1+1)));
if (vr#0) == 0 then (if s > 9 then (if (w#0)==(w#(n-2))
  then (if (w#(n-1)) == 1 then t=(w#0)*d1+1)));
t)
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