On the linearity defect of the residue field

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Linearity defect: definition

$(R, \mathfrak{m}, k)$= commutative local Noetherian ring; $\mathfrak{m} \neq 0$
M=finitely generated R-module;

$$R^{\mathrm{g}} = \oplus\ \mathfrak{m}^i/\mathfrak{m}^{i+1} \quad \text{and} \quad M^{\mathrm{g}} = \oplus\ \mathfrak{m}^i M/\mathfrak{m}^{i+1}M.$$

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Consider a minimal free resolution of M:

$$F = \qquad \cdots \to F_{n+1} \xrightarrow{d_n} F_n \to \cdots \to F_0 \to 0$$

and the filtration of F given by the subcomplexes:

$$\cdots \to F_{n+1} \to F_n \to \cdots \to F_i \to \mathfrak{m}F_{i-1} \to \mathfrak{m}^2 F_{i-2} \cdots \to \mathfrak{m}^i F_0 \to 0$$

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The associated graded complex is a complex of R^{g}-modules:

$$F^{\mathrm{g}} = \ \cdots \to {F_{n+1}}^{\mathrm{g}}(-n-1) \to {F_n}^{\mathrm{g}}(-n) \to \cdots \to {F_0}^{\mathrm{g}} \to 0$$

(Herzog and Iyengar): the *linearity defect* of M is the number:

$$\mathrm{ld}_R(M) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(F^{\mathrm{g}}) \neq 0\}.$$

Connections to regularity

- $\mathrm{ld}_R(M) = 0 \iff F^{\mathrm{g}}$ is a minimal free resolution of M^{g}.
  In this case, $\mathrm{reg}_{R^{\mathrm{g}}}(M^{\mathrm{g}}) = 0$.
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- $\mathrm{ld}_R(M) < \infty$ iff M has a syzygy which is Koszul.

Interpretation

If $i > 0$, let $\mu_i^n(M)$ denote the natural map

$$\operatorname{Tor}_i^R(\mathfrak{m}^{n+1}, M) \to \operatorname{Tor}_i^R(\mathfrak{m}^n, M)$$

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Theorem. *Let $i > 0$. Then:*

$$\mathrm{H}_i\left(F^{\mathsf{g}}\right) = 0 \iff \mu_i^n(M) = 0 = \mu_{i-1}^n(M) \text{ for all } n > 0.$$

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- $\mathrm{ld}_R(M) \le d \iff \mu_i^n(M) = 0$ for all $i \ge d$ and all $n > 0$.
- $\mathrm{ld}_R(M) = 0 \iff \mu_i^n(M) = 0$ for all i and n

The Graded case

When R is a standard graded k-algebra and M is a graded R-module, one can use the same definitions, with $\mathfrak{m} = R_{\geqslant 1}$.

Herzog and Iyengar: $\operatorname{ld}_R(M) < \infty \implies \operatorname{reg}_R(M) < \infty$

In particular: $\operatorname{ld}_R(k) < \infty \implies \operatorname{reg}_R(k) < \infty$, hence R is a Koszul algebra (Avramov and Peeva) and $\operatorname{ld}_R(k) = 0$.

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An analysis of the proof reveals that a weaker hypothesis suffices:

Proposition. $\mu^1_{\gg 0}(M) = 0 \implies \operatorname{reg}_R(M) < \infty$.
In particular, $\mu^1_{\gg 0}(k) = 0 \implies R$ *is Koszul.*

(Recall that $\mu^1_i \colon \operatorname{Tor}^R_i(\mathfrak{m}^2, M) \to \operatorname{Tor}^R_i(\mathfrak{m}, M)$.)

Questions

Back to the local case.

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- For any n: If $\mu^n_{\gg 0} = 0$ does it follow that $\mu^n = 0$?
- If $\operatorname{ld}_R(M) < \infty$ for every finitely generated R-module (R is *absolutely Koszul*), does it follow that R is Koszul?

The maps μ^1 and the Yoneda algebra

Think of μ_i^1 as $\operatorname{Ext}_R^{i+1}(k,k) \to \operatorname{Ext}_R^{i+1}(R/\mathfrak{m}^2,k)$
Set $E = \operatorname{Ext}_R(k,k)$, with Yoneda product.
Set $R^!$=the subalgebra of E generated by its elements of degree 1.

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- $\mu_i^1 = 0$.

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We have thus:

- $\mu_{>0}^1 = 0 \iff E = R^!$
- $\mu_{\geqslant s}^1 = 0 \iff E$ is generated/ $R^!$ by its elements of degree s.

In particular: **If R is a standard graded algebra and E is finitely generated over $R^!$, then $E = R^!$ and R is Koszul.**

Set $s(R) = \inf\{i \geq 1 \mid \mathfrak{a} \cap \mathfrak{n}^{i+2} \subseteq \mathfrak{n}\mathfrak{a}\}$
where $\widehat{R} = Q/\mathfrak{a}$ is a minimal regular presentation of R with $(Q, \mathfrak{n})$ regular local and $\mathfrak{a} \subseteq \mathfrak{n}^2$.

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Proposition. *The following hold:*

(a) *If* $\mu^1_{4n-1} = 0$ *for some positive integer* n*, then* $\mu^1_3 = \mu^1_1 = 0$

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For the proof: Use the fact that k has a minimal free resolution F with DG Γ algebra structure, obtained by adjoining variables.

Then think of μ^n_i as $H_{i+1}(F/\mathfrak{m}^2 F) \to H_{i+1}(F/\mathfrak{m}F)$.
Thus $\mu^n_i = 0$ means: If $dx \in \mathfrak{m}^2 F_i$, then $x \in \mathfrak{m}F_{i+1}$.

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$$
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\mu^1_{>0} = 0 & \Longleftrightarrow & E = R^! \\
\Downarrow & & \Downarrow \\
\mu^1_{\geqslant s} = 0 & \Longleftrightarrow & \begin{array}{c} E \text{ is generated over } R^! \\ \text{by elements of degree s} \end{array} \\
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Complete intersection rings

Assume R is a complete intersection: $\widehat{R} = Q/$(regular sequence), with Q regular local. For these rings: $s(R) = 1 \iff E = R^!$.

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Proposition. *If $\operatorname{ld}_R(k) < \infty$, then $E = R^!$.*

Under a stronger hypothesis, we obtain a stronger conclusion:

Theorem

The following statements are equivalent:

(a) $\operatorname{ld}_R k = 0$ *(R is Koszul)*

(b) *R has minimal multiplicity.*

Furthermore, if R^{g} is Cohen-Macaulay, then they are also equivalent to

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Proof of (c) $\Longrightarrow$ (b): Reduce first to the Artinian case. Then, a length count.

Artinian rings

Theorem

Assume R is Artinian with $\mathfrak{m}^{n+1} = 0$. If $\mu_{\gg 0}^{n-1} = 0$, then $\mu_{>0}^{n-1} = 0$.

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Corollary

If R is Golod and R^{g} is Cohen-Macaulay, then the following statements are equivalent:

(a) $\operatorname{ld}_R k = 0$

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(c) *R has minimal multiplicity ($codim R = e(R) - 1$)*

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If R is Golod and R^{g} is Cohen-Macaulay, then the following statements are equivalent:

(a) $\mathrm{ld}_R\, k = 0$

(b) $\mathrm{ld}_R\, k < \infty$

(c) R *has minimal multiplicity* ($codim R = e(R) - 1$)

The corollary follows from the Theorem, using the fact that an Artinian Golod ring does not have any non-zero small ideals.

Proof.

Let $j > 0$ and let i large enough so that $\mu_{i+j}^{n-1} = 0$, thus the natural map $\mathrm{Ext}_R^{i+j}(\mathfrak{m}^{n-1}, k) \to \mathrm{Ext}_R^{i+j}(\mathfrak{m}^n, k)$ is zero.

$$\begin{array}{ccccc} E^i \otimes \mathrm{Ext}_R^j(\mathfrak{m}^{n-1}, k) & \longrightarrow & E^i \otimes \mathrm{Ext}_R^j(\mathfrak{m}^n, k) & \xrightarrow{\cong} & E^i \otimes E^j \otimes \mathrm{Hom}_R(\mathfrak{m}^n, k) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ext}_R^{i+j}(\mathfrak{m}^{n-1}, k) & \xrightarrow{0} & \mathrm{Ext}_R^{i+j}(\mathfrak{m}^n, k) & \xrightarrow{\cong} & E^{i+j} \otimes \mathrm{Hom}_R(\mathfrak{m}^n, k) \end{array}$$

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Assume that $\mu_j^{n-1} \neq 0$: $\operatorname{Ext}_R^j(\mathfrak{m}^{n-1}, k) \xrightarrow{\neq 0} \operatorname{Ext}_R^j(\mathfrak{m}^n, k)$.

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$\exists$ thus an element in $\operatorname{Ext}_R^j(\mathfrak{m}^{n-1}, k)$ whose image θ in $\operatorname{Ext}_R^j(\mathfrak{m}^n, k)$ is non-zero.

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Commutativity of the squares yields: $\exists\ \alpha \in E^j$ non-zero such that $\varphi\alpha = 0$ for all $\varphi \in E^i$. Thus the element α of E is anihhilated by all elements of E of sufficienlty large degree. However, this is a contradiction, according to a result of Martsinkovski. □