Bounding projective dimension and regularity

Alexandra Seceleanu
joint with J. Beder, J. McCullough, L. Nuñez, B. Snapp, B. Stone

October 15, 2011
AMS Sectional Meeting, Lincoln
Given an ideal $I \subset S = K[X_1, \ldots, X_n]$ there are two measures of the computational complexity of finding the resolution of $S/I$:

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- projective dimension = "width" of the Betti table (last nonzero column);
- regularity = "height" of the Betti table (last non-zero row).
Stillman’s Question

Question (Stillman)

Is there a bound, independent of \(n\), on the projective dimension of ideals in \(S = K[X_1, \ldots, X_n]\) which are generated by \(N\) homogeneous polynomials of given degrees \(d_1, \ldots, d_N\)?

Remark

Hilbert’s Syzygy Theorem guarantees \(\text{pd}(S/I) \leq n\), but we seek a bound independent of \(n\).
Stillman’s Question

Known cases:

- If \( I = (m_1, \ldots, m_N) \) is a monomial ideal, then \( pd(S/I) \leq N \) by the Taylor resolution. Note that \( N \) does not work in general.

- If \( I = (f, g, h) \) with \( f, g, h \) quadrics, then \( pd(S/I) \leq 4 \) by Eisenbud-Huneke (unpublished). This bound is tight.

- If \( I = (f, g, h) \) with \( f, g, h \) cubics, then \( pd(S/I) \leq 36 \) by Engheta. The tight bound in this case is likely to be 5.
A bound for ideals of quadrics

Two ideas in pursuing this question:
1. look at ideals generated in small degrees (quadrics, cubics)
2. limit the number of generators (three-generated ideals)

Theorem (Ananyan-Hochster, 2011)
Let $S = K[x_1, \ldots, x_n]$, let $F_1, \ldots, F_N$ be polynomials of degree at most 2 and $I = (F_1, \ldots, F_N)$. Then there is a function $C(N)$ such that $I$ is contained in the $K$-subalgebra of $S$ generated by a regular sequence of at most $C(N)$ forms of degree at most 2. Consequently the projective dimension of $S/I$ is at most $C(N)$.

Remark
The asymptotic growth of $C(N)$ is of order $2N^{2^N}$. 
Three-generated ideals

Theorem (Burch-Kohn, 1968)

For any \( n \in \mathbb{N} \), there is a three-generated ideal \( I = (f, g, h) \) in a polynomial ring \( S = K[x_1, \ldots, x_{2n}] \) with \( pd(S/I) = n \).

Remark

Engheta computed the degrees of the three generators to be \( n - 2, n - 2, 2n - 2 \)
Three-generated ideals

Theorem (Bruns, 1976)

Any resolution is the resolution of a three-generated ideal.

Remark (Nguyen, Niu, Sanyal, Torrance, Witt, Zhang)

Note that degrees of the generators of the brunsification of an ideal grow, but can be controlled. e.g. brunsification of \((X_1^d, \ldots, X_n^d)\) yields three generators of degree at most \(d(n - 2)^2\).
Y. Zhang's Question

Question (Y. Zhang)

Assume $I = (f_1, \ldots, f_N)$ is an ideal of $S = K[X_1, \ldots, X_n]$. Is it true that $pd(S/I) \leq \sum_{i=0}^{N} \deg f_i$?

The following constructions show this bound is (much) too small.
McCullough’s ideals with large projective dimension

Fix integers $m, n, d$ such that $m \geq 1$, $n \geq 0$ and $d \geq 2$.

Let $Z_1, \ldots, Z_p$ be the $\frac{(m+d-2)!}{(m-1)!(d-1)!}$ monomials of degree $d - 1$ in $X_1, \ldots, X_m$.

Example

$$S = k[X_1, \ldots, X_n, Y_{1,1}, \ldots, Y_{p,n}]$$

$$I_{m,n,d} = \left( X_1^d, \ldots, X_n^d, \sum_{i=0}^{p} Z_j Y_{i,1}, \ldots, \sum_{j=0}^{p} Z_j Y_{j,n} \right)$$

is generated by $m + n$ degree $d$ generators
Theorem (McCullough, 2011)

\[ pd(R/I_{m,n,d}) = m + np = m + n \frac{(m + d - 2)!}{(m - 1)!(d - 1)!}. \]

Proof sketch:

Show \( \text{depth}(R/I_{m,n,d}) = 0 \) and apply Auslander-Buchsbaum.
Example: \( I_{3,4,2} \)

\[
S = K[X_1, \ldots, X_m, Y_{1,1}, \ldots, Y_{3,4}]
\]

\[
I = (X_1^2, X_2^2, X_3^2, X_1 Y_{1,1} + X_2 Y_{2,1} + X_3 Y_{3,1}, X_1 Y_{1,2} + X_2 Y_{2,2} + X_3 Y_{3,2},
X_1 Y_{1,3} + X_2 Y_{2,3} + X_3 Y_{3,3}, X_1 Y_{1,4} + X_2 Y_{2,4} + X_3 Y_{3,4})
\]

\( I \) has 7 quadratic generators and \( pd(S/I) = \# \) variables = 15 > 7 \( \cdot \) 2.

The answer to Zheng's question is negative.
A new family

Example (The ideal $I = I_{2,(2,2,2)}$)

$$A_0 = \{(0,0,0)\}, A_1 = \{(1,0,0)\}, A_2 = \{(1,1,0)\},$$
$$A_3 = \{(1,1,2), (1,1,1), (1,1,0)\}.$$

$$f = x^{(0,0,0)} x_{1,1} x_{1,2}^2 + x^{(0,0,0)} x_{2,1} x_{2,2}^2 + x^{(1,0,0)} x_{1,2} x_{1,3} + x^{(1,0,0)} x_{2,2} x_{2,3}$$
$$+ x y^{(1,1,2)} + x y^{(1,1,1)} + x y^{(1,1,0)}$$

$$= x_{1,1} x_{1,2}^2 + x_{2,1} x_{2,2}^2 + x_{1,1} x_{2,1} x_{1,2} x_{1,3} + x_{1,1} x_{2,1} x_{2,2} x_{2,3}$$
$$+ x_{1,1} x_{2,1} x_{1,2} x_{2,2} x_{1,3} y^{(1,1,2)} + x_{1,1} x_{2,1} x_{1,2} x_{2,2} x_{1,3} x_{2,3} y^{(1,1,1)}$$
$$+ x_{1,1} x_{2,1} x_{1,2} x_{2,2} x_{2,3} y^{(1,1,0)}.$$

Finally, the ideal $I = \left(x_{1,1}^7, x_{2,1}^7, f\right)$. 
Larger projective dimension

Fix $g \geq 2, m_n \geq 0, m_{n-1} \geq 1, m_i \geq 2$ for $1 \leq i \leq n - 2$.

$$I = I_{g,(m_1,\ldots,m_{n-1})}$$

Theorem (Beder, McCullough, Nuñez, S-, Snapp, Stone)

$$pd(R/I) = \prod_{i=1}^{n-1} \left( \frac{(m_i + g - 1)!}{(g - 1)!(m_i)!} - g \right) \left( \frac{(m_n + g - 1)!}{(g - 1)!(m_n)!} \right) + gn.$$  

Proof: Count the variables: $g \times n \times X$ variables and $|\mathcal{A}_n| Y$ variables.
Corollary (Beder, McCullough, Nuñez, S-, Snapp, Stone)

Over any field \( K \) and for any positive integer \( p \), there exists an ideal \( I \) in a polynomial ring \( R \) over \( K \) with three homogeneous generators in degree \( p^2 \) such that \( pd(R/I) \geq p^{p-1} \).

Proof:

\[ I = I_{2,(p+1,...,p+1,0)} \cdot (p-1 \text{ times}) \]

Corollary (Beder, McCullough, Nuñez, S-, Snapp, Stone)

Over any field \( K \) and for any positive integer \( p \), there exists an ideal \( I \) in a polynomial ring \( R \) over \( K \) with \( 2p + 1 \) homogeneous generators in degree \( 2p + 1 \) such that \( pd(R/I) \geq p^{2p} \).

Proof:

\[ I = I_{2p,(2,2,2,...,2)} \cdot (p \text{ times}) \]
$I = I_{2,(4,1)}$

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Alexandra Seceleanu (UNL)  
Bounding projective dimension and regularity  
Oct 15, 2011
\( I = I_{2, (2,1,2)} \)

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Stilman’s Question - Regularity Version

Question (Stillman)

Is there a bound, independent of $n$, on the regularity of ideals in $S = K[X_1, \ldots, X_n]$ which are generated by $N$ homogeneous polynomials of given degrees $d_1, \ldots, d_N$?

Caviglia proved:

the regularity question $\iff$ the projective dimension question.

Caution: this does not mean the bounds will be the same.
Caviglia’s subfamily

Let
\[ C_d = (w^d, x^d, wy^{d-1} + xz^{d-1}) \subset S = K[w, x, y, z] \]

Caviglia showed \( \text{reg}(S/C_d) = d^2 - 1 \).

\( C_d \) is a subfamily of the new family: \( C_d = I_{2,(1,d-2)} \)

Question

What is the asymptotic growth of \( \text{reg}(I_{2,(2,1,d)}) \)?
Conjectures

Let

\[ C_d = (w^d, x^d, wy^{d-1} + xz^{d-1}) \subset S = K[w, x, y, z] \]

Caviglia showed \( \text{reg}(S/C_d) = d^2 - 1 \).

\( C_d \) is a subfamily of the new family: \( C_d = I_{2,(1,d-2)} \)

Conjecture

We believe \( \text{reg}(I_{2,(2,1,d)}) \) exhibits cubic growth in \( d \).

We believe \( \text{reg}(I_{2,(2,2,2,\ldots,2,1,d)}) \) grows asymptotically as \( d^{p+2} \).

\( p \) times
