Classifying $n \leq 8$ points in $\mathbf{P}^2$

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Betti number results for $n = 6$ points: joint with Elena Guardo.

Classification of configuration types for $n \leq 8$ points: joint with Tony Geramita and Juan Migliore.

Some General Problems

Let $I \subset R = k[x, y, z]$ be ideal of reduced points $p_1, \ldots, p_n \in \mathbf{P}^2$.

Problem 1: (cf. Evans-Richert) Given Hilbert function of $I^{(m)}$, what can you say about graded Betti numbers of $I^{(m)}$?

Problem 2: (Geramita-Migliore-Sabourin) Given Hilbert function of I, what can you say about Hilbert function of $I^{(2)}$?

Problem 3: What functions can be Hilbert functions of ideals $I^{(m)}$?

Problem 1: Given Hilbert function of $I^{(m)}$, what can you say about graded Betti numbers of $I^{(m)}$?

Complete Answers for points on conic or any $n \leq 6$ points.

based on Catalisano (1991: smooth conic)/Harbourne (1998: any conic), and Guardo-Harbourne (2005: $n \leq 6$).

Problem 2: Given Hilbert function of I, what can you say about Hilbert function of $I^{(2)}$?
Problem 3: What functions can be Hilbert functions of ideals $I^{(m)}$, $m > 0$?

Complete Answers for $n \leq 8$ points (any m) or $n = 9$ for $m = 2$ (Geramita-Harbourne-Migliore 2006).

Examples:

(1) Say $R/I^{(m)}$ has Hilbert function:

`1 3 6 10 14 16 17 18 18 18 18 18.`

Exactly what can the graded Betti numbers be? **Answer**:

deg	4	5	6	7	8	9
gen	1	2	0	0	1	0
syz	0	0	1	1	0	1

OR

deg	4	5	6	7	8	9
gen	1	2	1	0	1	0
syz	0	0	2	1	0	1

(2) Say R/I has Hilbert function: `1 3 5 6 6 6 6 6.`

What functions occur as Hilbert functions of $R/I^{(2)}$? **Answer**:

```
1 3 6 10 14 16 17 18 18 18 18 18
1 3 6 10 14 16 17 18 18 18 18 18
1 3 6 10 14 17 18 18 18 18
1 3 6 10 14 17 18 18 18 18
```

Fat points: Review

Given points $p_1, \dots, p_n \in \mathbf{P}^2$ and integers $m_i \geq 0$.

Notation: $Z = m_1 p_1 + \cdots + m_n p_n$ denotes the subscheme, a *fat points* subscheme, of $\mathbf{P}^2$ defined by ideal

$$I(Z) = I(p_1)^{m_1} \cap \cdots \cap I(p_n)^{m_n} \subset R = k[x, y, z].$$

Hilbert function of I(Z):

$$h(I(Z))(t) = \dim_k I(Z)_t$$

Graded Betti numbers γ_i and σ_j of I(Z) in minimal free resolution:

$$0 \to \bigoplus_{j \geq 0} R[-j]^{\sigma_j} \to \bigoplus_{i \geq 0} R[-i]^{\gamma_i} \to R \to R/I \to 0$$

Definition: Hilbert function equivalence

Ordered point sets $p_1, \ldots, p_n$ and $p'_1, \ldots, p'_n$ are *Hilbert function equivalent* if

$$Z = m_1 p_1 + \cdots + m_n p_n \text{ and } Z' = m_1 p'_1 + \cdots + m_n p'_n$$

have the same Hilbert functions for every choice of the m_i.

A Hilbert function equivalence class is a **Configuration Type**.

Table summarizing results

n	1	2	3	4	5	6	7	8	9	10	etc.
# CT's	1	1	2	3	5	11	29	143	∞	∞	∞

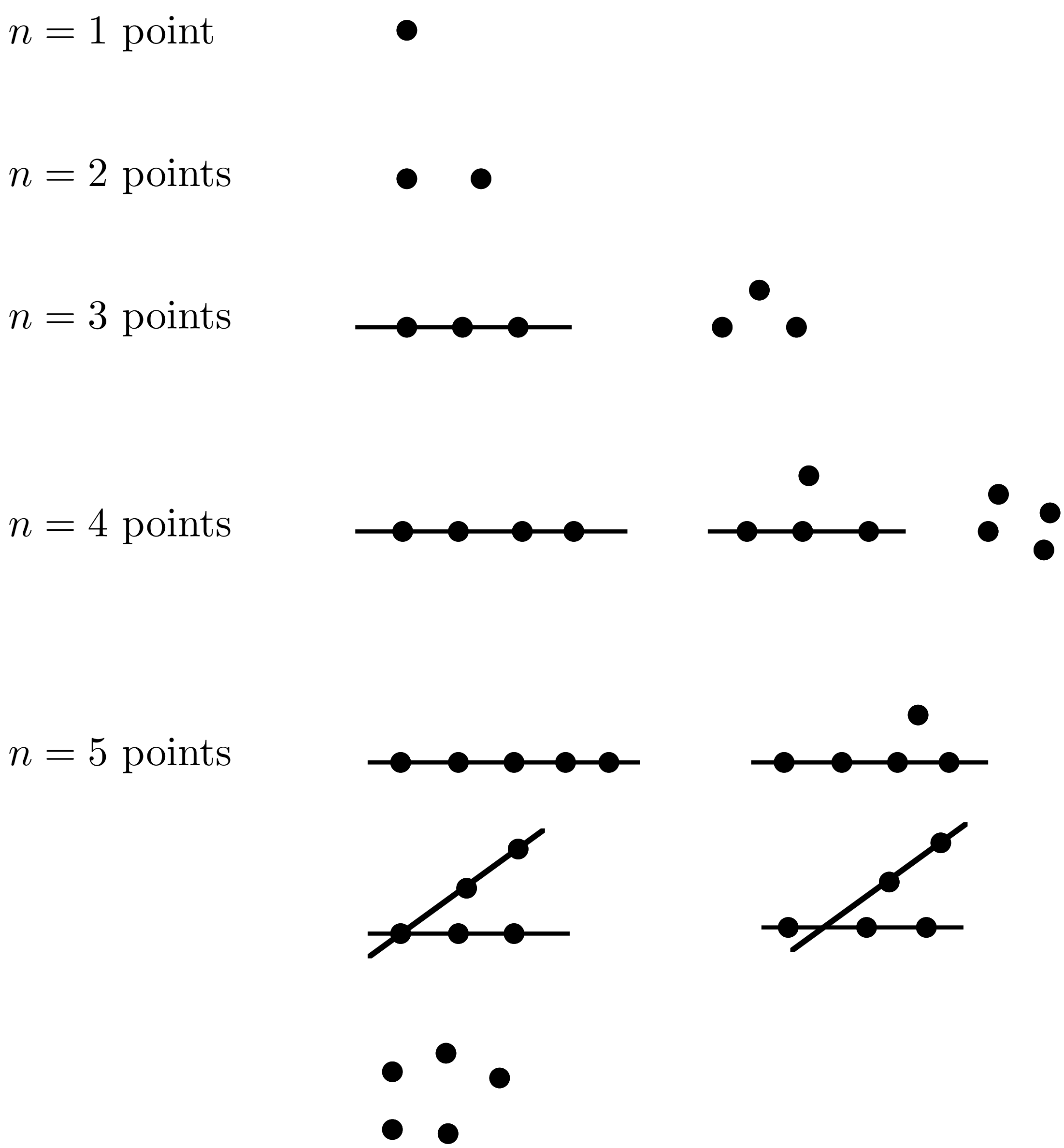
Method (GHM 2006):

- Reinterpret as a problem on algebraic surfaces

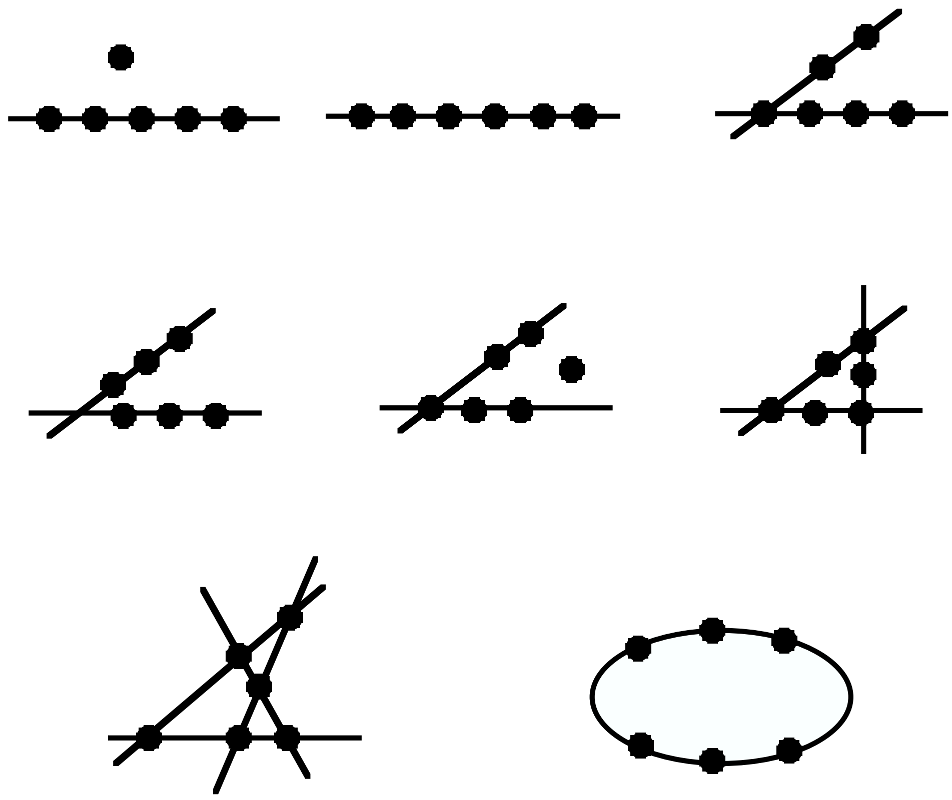
- Adjunction formula reduces problem to combinatorial matroid-like formal classification

- Must verify representability of the formal possibilities

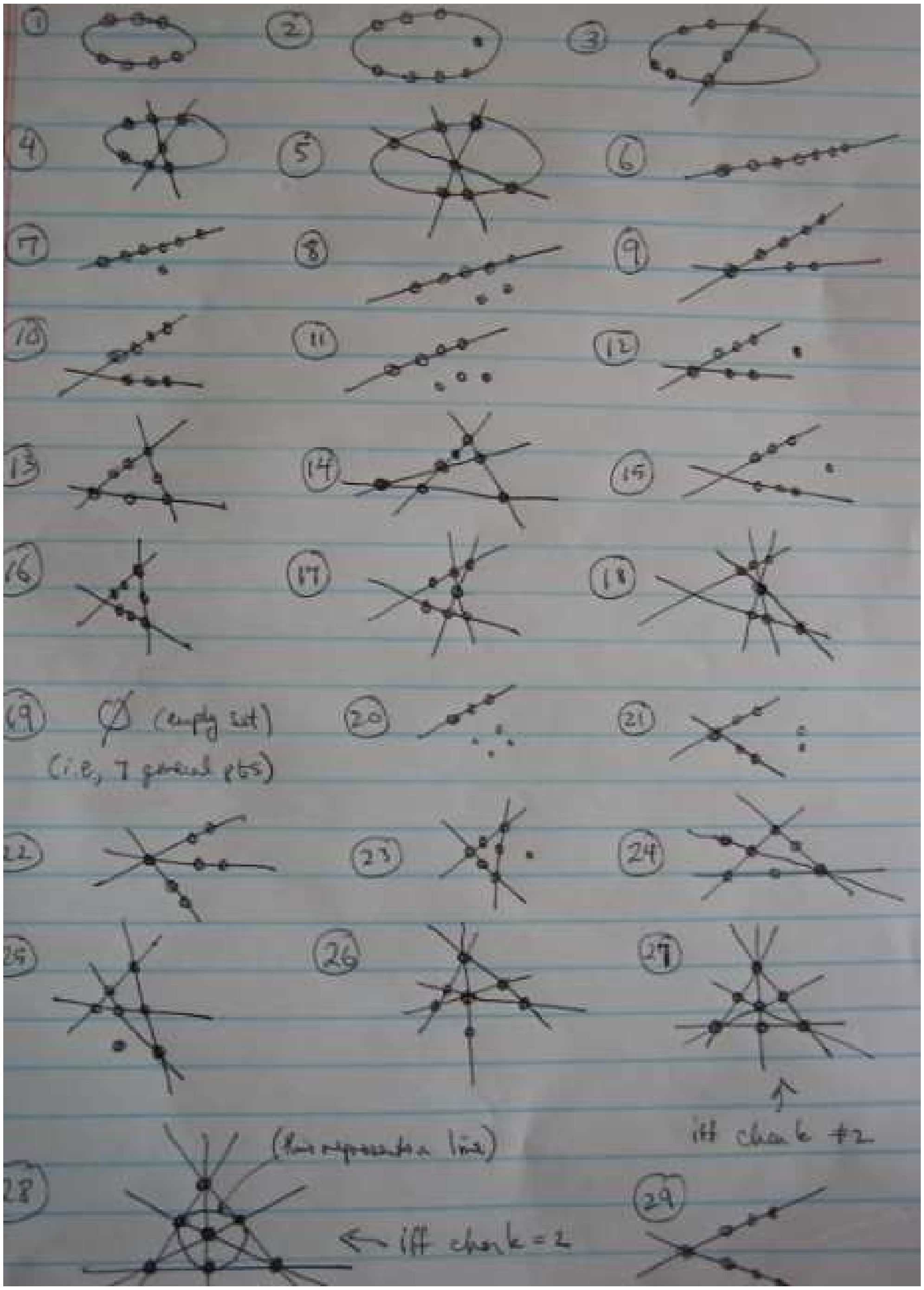
- The occurrence of certain CT's can depend on the characteristic of ground field, k.



Configuration Types of $n \leq 5$ points in $\mathbf{P}^2$



Configuration Types of $n = 6$ points in $\mathbf{P}^2$



Configuration Types of $n = 7$ points in $\mathbf{P}^2$

Configuration Types of $n = 8$ points in $\mathbf{P}^2$

1 empty
2 1: abc
3 1: abc, def
4 1: abc, ade
5 1: abc, ade, afg
6 1: abc, ade, bdf
7 1: abc, ade, bfg
8 1: abc, ade, fgh
9 1: abc, ade, bdf, cgh
10 1: abc, ade, bdf, ceg
11 1: abc, ade, bdf, cef
12 1: abc, ade, bfg, dfh
13 1: abc, ade, afg, bdf
14 1: abc, ade, afg, bdh
15 1: abc, ade, afg, bdf, ceg
16 1: abc, ade, afg, bdf, beg
17 1: abc, ade, afg, bdf, ceh
18 1: abc, ade, afg, bdf, beh
19 1: abc, ade, afg, bdh, ceh
20 1: abc, ade, afg, bdh, cfh
21 1: abc, ade, bdf, cgh, efg
22 1: abc, ade, afg, bdf, ceg, beh
23 1: abc, ade, afg, bdf, beg, cdg
24 1: abc, ade, afg, bdf, beg, dgh
25 1: abc, ade, afg, bdf, beg, cdh
26 1: abc, ade, afg, bdf, ceh, bgh
27 1: abc, ade, afg, bdh, cfh, egh
28 1: abc, ade, afg, bdf, ceg, beh, cfh
29 1: abc, ade, afg, bdf, ceg, beh, cdh
30 1: abc, ade, afg, bdf, beg, cdg, ceh
31 1: abc, ade, afg, bdf, ceg, beh, cfh, dgh
32 3: abcdefgh
33 1: abc, 3: abcdefgh
34 1: abc, def, 3: abcdefgh
35 1: abc, ade, 3: abcdefgh
36 1: abc, ade, afg, 3: abcdefgh
37 1: abc, ade, bdf, 3: abcdefgh
38 1: abc, ade, bfg, 3: abcdefgh
39 1: abc, ade, bdf, ceg, 3: abcdefgh
40 1: abc, ade, bdf, cef, 3: abcdefgh
41 1: abc, ade, afg, bdf, 3: abcdefgh
42 1: abc, ade, afg, bdf, ceg, 3: abcdefgh
43 1: abc, ade, afg, bdf, beg, 3: abcdefgh
44 1: abc, ade, afg, bdf, beg, cdg, cef, 3: abcdefgh
45 2: abcdef, 3: abcdefgh
46 1: abc, 2: bcdefg, 3: abcdefgh
47 1: abc, ade, 2: bcdefg, 3: abcdefgh
48 1: abc, ade, afg, 2: bcdefg, 3: abcdefgh
49 2: abcdef
50 2: abcdef, abcdgh
51 2: abcdef, abcdgh, abefgh
52 2: abcdef, abcdgh, abefgh, cdefgh
53 1: abc, ade, fgh, 2: bcdegh
54 1: abc, ade, bdf, cgh, 2: abefgh
55 1: abc, ade, afg, bdf, 2: cdefgh
56 1: abc, ade, afg, bdf, beg, 2: cdefgh
57 1: abc, ade, afg, bdf, beh, 2: cdefgh
58 1: abc, ade, afg, bdh, ceh, 2: bcdefg
59 1: abc, ade, afg, bdh, cfh, 2: bcdefg
60 1: abc, ade, afg, bdh, cfh, egh, 2: bcdefg
61 1: abc, 2: cdefgh
62 1: abc, ade, 2: bcdegh
63 1: abc, ade, afg, 2: bcdefg
64 1: abc, ade, bdf, 2: cdefgh
65 1: abc, ade, bfg, 2: cdefgh
66 1: abc, ade, afg, bdh, 2: cdefgh
67 1: abc, 2: bcdefg
68 1: abc, ade, 2: cdefgh
69 1: abc, ade, afg, 2: cdefgh
70 1: abc, ade, bdf, 2: bcdegh
71 1: abc, ade, bfg, 2: bcdegh
72 1: abc, ade, afg, bdh, 2: bcdefg
73 1: abc, ade, 2: acefgh
74 1: abc, def, 2: bcefgh
75 1: abc, ade, bdf, ceg, 2: acdfgh
76 1: abc, ade, bdf, cef, 2: bcdegh
77 1: abc, ade, bfg, dfh, 2: bcdegh
78 1: abc, 2: cdefgh, abefgh
79 1: abc, ade, 2: bcdegh, acefgh
80 1: abc, ade, bdf, 2: bcdegh, acdfgh
81 1: abc, 2: acdefg, abefgh
82 1: abc, ade, 2: bdefgh, acefgh
83 1: abc, ade, bdf, 2: cdefgh, abefgh
84 1: abc, ade, 2: acefgh, abdfgh
85 1: abc, def, 2: bcefgh, acdfgh
86 1: abc, ade, bfg, 2: bcdegh, acefgh
87 1: abc, ade, bdf, ceg, 2: acdfgh, abefgh
88 1: abc, ade, bdf, cef, 2: bcdegh, acdfgh
89 1: abc, ade, bfg, dfh, 2: bcdegh, acefgh
90 1: abc, 2: bcdfgh, acdefg, abefgh
91 1: abc, def, 2: bcefgh, acdfgh, abdegh
92 1: abc, ade, 2: bcdegh, acefgh, abdfgh
93 1: abc, ade, bdf, 2: bcdegh, acdfgh, abefgh
94 1: abc, ade, afgh, 2: bcdegh
95 1: abc, defg
96 1: abc, ade, afgh
97 1: abc, ade, bdf, afgh
98 1: abc, ade, afg, bdf, cegh
99 1: abc, adeh, 2: bcdefg
100 1: abc, ade, bdf, afgh, 2: bcdegh
101 1: abc, adef
102 1: abc, ade, bdfg
103 1: abc, ade, bdf, cegh
104 1: abc, ade, afg, bdf, begh
105 1: abc, ade, bdfg, 2: acefgh
106 1: abc, ade, bfgh
107 1: abc, ade, bdf, cefg
108 1: abc, def, adgh, 2: bcefgh
109 1: abc, ade, bdf, cef, afgh, 2: bcdegh
110 1: abcd
111 1: abc, ade, afg, bdfh
112 1: abcd, efgh
113 1: abcd, aefg
114 1: abc, adef, bdgh
115 1: abc, ade, bdfg, cefh
116 1: abc, ade, afg, bdfh, cegh
117 1: abgh, 2: abcdef
118 1: efgh, 2: abcdef, abcdgh
119 1: abc, def, adgh
120 1: abc, ade, bfg, cdfh
121 1: abc, ade, bdf, ceg, afgh
122 1: abc, ade, bdf, cef, afgh
123 1: abc, ade, bfg, dfh, cegh
124 1: abc, ade, afg, bdh, cefh
125 1: abc, ade, afg, bdf, beg, cdgh
126 1: abc, ade, afg, bdf, beh, cdgh
127 1: abc, ade, afg, bdf, beg, cdg, cefh
128 1: abc, ade, afg, bdf, beg, dgh, cefh
129 1: abcde
130 1: abcde, afgh
131 1: abcde, fgh
132 1: abcde, afg
133 1: abcde, afg, bfh
134 1: abcde, afg, bfh, cgh
135 1: abcdef
136 1: abcdef, agh
137 1: abcdefg
138 1: abcdefgh
139 2: abcdefg
140 1: abh, 2: abcdefg
141 1: abh, cdh, 2: abcdefg
142 1: abh, cdh, efh, 2: abcdefg
143 2: abcdefgh

Theorem: Given $p_1, \ldots, p_n \in \mathbf{P}^2$ and $Z = m_1p_1 + \cdots + m_np_n$.

(1) $n \leq 8$: The configuration type and multiplicities m_i determine $h(I(Z))(t)$ explicitly for each t.

[Computing $h(I(Z))(t)$ involves just dot products of integer vectors (corresponding to Bezout's Theorem).]

(2) $n \leq 6$: The configuration type and the multiplicities m_i also determine the graded Betti numbers of $I(Z)$ explicitly.

[Computing them again involves just dot products of integer vectors (corresponding to Bezout's Theorem).]

Basic Principle Underlying Proof:

After accounting for curves in base locus of $I(Z)_t$, the expected behavior is the actual behavior.

Note:

The needed calculations are easy enough to do by hand, but they are somewhat tedious. They've been implemented as web forms at:

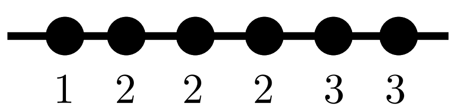
http://www.math.unl.edu/~bharbourne1/FatPointAlgorithms.html

Computing Hilbert functions:

- Configuration type determines finite list of possible base curves
- Use Bezout to check which are base curves for $I(Z)_t$
- take scheme Z' residual to Z with respect to base curves
- Z' is a fat point scheme, free of base curves in residual degree t'
- [Harbourne, mid 90s] $I(Z')_{t'}$ fixed component free $\Longrightarrow$

$$h(I(Z))(t) = h(I(Z'))(t') = \binom{t'+2}{2} - \binom{m'_1+1}{2} - \cdots - \binom{m'_n+1}{2}$$

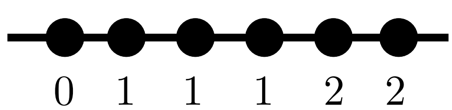
Example: Compute $h(I(Z))(8)$ for:

Z: 

$\ell = \{L = 0\}$

- Configuration Type $\Longrightarrow$ ℓ is the only possible base curve
- Bezout $\Longrightarrow$ ℓ is in fact base curve of $I(Z)_8$:

$$8 < 1 + 2 + 2 + 2 + 3 + 3 = 13$$

- residual Z': 

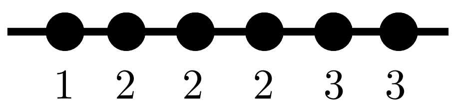
- residual degree $t' = 8 - \text{degree}(\ell) = 7$
- Apply formula: $h(I(Z))(8) = h(I(Z'))(7) = 27$

Computing graded Betti numbers (when $n \leq 6$):

- $\gamma_{t+1} = \dim \operatorname{cok}(I(Z)_t \otimes R_1 \xrightarrow{\mu} I(Z)_{t+1})$:
- μ has maximal rank if $I(Z)_t$ is base curve free [Guardo–Harbourne, 2005]

Example: Compute $\gamma_8 = \dim \operatorname{cok}(I(Z)_7 \otimes R_1 \xrightarrow{\mu} I(Z)_8)$ for

Z:



$\ell = \{L = 0\}$

- Note: $R_1 = \operatorname{Span}(x, y, z)$
- Bezout $\Longrightarrow$ base curve of $I(Z)_7$ is 2ℓ
- Residuals Z'' and $t'' = 5$; i.e., $I(Z)_7 = L^2 \cdot I(Z'')_5$
- $h(I(Z))(7) = h(I(Z''))(5) = \binom{5+2}{2} - 0 - 0 - 0 - 1 - 1 = 19$
- $h(I(Z''))(6) = \binom{6+2}{2} - 0 - 0 - 0 - 1 - 1 = 26$
- μ factors as μ'' (of maximal rank) and inclusion:

$$I(Z)_7 \otimes R_1 = L^2 \cdot I(Z'')_5 \otimes R_1 \xrightarrow{\mu''} \underbrace{L^2 \cdot I(Z'')_6 \subset I(Z)_8}$$

maximal rank: dim cok $= 0$,
since $3(19) > 26$

difference in dimensions $= 27 - 26 = 1$

- Thus $\gamma_8 = 0 + 1 = 1$

Corollary: Consider the ideal I of n points in $\mathbf{P}^2$.

For each $n \leq 7$, there is an m_n such that the Hilbert function of $I^{(m_n)}$ determines the Hilbert function of $I^{(m)}$ for all $m \geq 0$:

n	1	2	3	4	5	6	7	8
m_n	1	1	1	2	3	3	7	22?

Proof of Corollary: Just check that the Hilbert function of $I^{(m_n)}$ distinguishes the configuration types.

For example: If $n = 6$, the Hilbert function of $I^{(3)}$ determines the graded Betti numbers of $I^{(m)}$ for all $m \geq 1$.

Example showing run times (Gröbner bases vs. Theorem):

```
via Macaulay2 (6 random points of multiplicities: 30 13 23 33 3 13)

   50: . 10  6
   51: .  6  8
   52: .  6  6
   53: .  5  5
   54: .  5  5
   55: .  5  5
   56: .  1  2
   57: .  1  1
   58: .  1  1
   59: .  1  1
   60: .  1  1
   61: .  1  1
   62: .  1  1
     -- used 154.27 seconds
##############################################
via web form (run locally):
..........................................
Configuration 1: Z = 30 13 23 33 3 13
(i.e., 6 general points of multiplicities: 30 13 23 33 3 13)
..........................................

deg    hilb_I   gens    syz     hilb_Z
51        10      10      0       1368
52        30       6      6       1401
53        58       6      8       1427
54        93       5      6       1447
55       135       5      5       1461
56       184       5      5       1469
57       236       1      5       1475
58       290       1      2       1480
59       346       1      1       1484
60       404       1      1       1487
61       464       1      1       1489
62       526       1      1       1490
63       590       1      1       1490
64       655       0      1       1490
65       721       0      0       1490
66       788       0      0       1490
..........................................
 -- used 3.25 seconds
```