The Ideal Generation Problem for Fat Points

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Abstract: This paper is concerned with the problem of determining up to graded isomorphism the modules in a minimal free resolution of a fat point subscheme $Z = m_1p_1 + \cdots + m_rp_r \subset \mathbf{P}^2$ for general points $p_1, \ldots, p_r$.

I. Introduction

We always work over an arbitrary algebraically closed field k. This paper is concerned with determining the number $\nu_t(I(Z))$ of elements in each degree t of any minimal set of homogeneous generators in the ideal $I(Z) \subset k[\mathbf{P}^2]$ defining a fat point subscheme $Z = m_1p_1 + \cdots + m_rp_r \subset \mathbf{P}^2$, where $p_1, \ldots, p_r \in \mathbf{P}^2$ are general. Given the Hilbert function of $I(Z)$, this is equivalent up to graded isomorphism to determining the modules in a minimal free resolution of $I(Z)$.

As discussed further below, the above problem has been solved for subschemes $Z = p_1 + \cdots + p_r \subset \mathbf{P}^2$, for general points p_i. The solution rests on showing in such cases that $I(Z)$ has the maximal rank property: given a graded ideal I in a polynomial ring R graded in the usual way by degree, we say that I has the *maximal rank property* if the multiplication maps $\mu_t(I) : I_t \otimes R_1 \to I_{t+1}$ have maximal rank (i.e., are injective or surjective) for every t (where the subscript t denotes homogeneous components of degree t).

Since fat point subschemes commonly fail to have the maximal rank property, it has been unclear what sort of answer to the general problem can be expected. In this paper we suggest an asymptotic solution. In particular, fixing points $p_1, \ldots, p_r \in \mathbf{P}^2$, we define an equivalence relation, Cremona equivalence, on fat point subschemes $Z = m_1p_1 + \cdots + m_rp_r$, and, if the points are strongly nonspecial (see Section III) and Z is expectedly good (a property defined below giving control over the Hilbert function of $I(Z)$, and which holds in all known cases for general points p_i), we show in Corollary III.5 for all but finitely many subschemes Z in each Cremona equivalence class that $I(Z)$ satisfies the maximal rank property.

There nevertheless remains the problem of understanding failures of the maximal rank property. Sometimes this is easy. For example, given a graded ideal $I \subset R$, define $\alpha(I)$ to be the least degree among nonzero homogeneous elements of I, and define $\beta(I)$ to be the least degree t such that the elements of I_t have no nontrivial common divisor. It is easy to see that μ_t cannot be injective for $t > \alpha$ and cannot be surjective for $t = \beta - 1$, so having $\alpha < \beta - 1$ guarantees that $\mu_{\beta-1}$ fails to have maximal rank and thus that I does not have the maximal rank property. On the other hand, failures of μ_β to have maximal rank are more mysterious, and, in fact, by Lemma II.2 and Lemma II.5 the general problem of determining numbers of generators for expectedly good fat point subschemes reduces to determining the rank of μ_β.

For an expectedly good fat point subscheme Z with $\alpha < \beta$, Fitchett [Fi] shows that the greatest common divisor of $I(Z)_\alpha$ determines the rank of μ_β. This work gives a geometric explanation for the possible failure of maximal rank of μ_β in the case that $\beta > \alpha$, in addition to determining bounds on the rank of μ_β.

What is still lacking is a general understanding of why μ_β could fail to have maximal rank in the case that $\alpha = \beta$. Our result Corollary III.5 on Cremona equivalence actually shows in the expectedly good case not only that the maximal rank property holds asymptotically but that $\alpha = \beta$ holds asymptotically as well.

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However, μ_β can fail to have maximal rank even if $\alpha = \beta$, and we study this phenomenon in the case of *uniform* fat point subschemes (i.e., subschemes $Z = m(p_1 + \cdots + p_r)$). For example, from our results in Section IV it follows that:

Corollary I.1: *Let $p_1, \ldots, p_r$ be $r \leq 9$ general points of $\mathbf{P}^2$ and let $I = I(m(p_1+\cdots+p_r))$. Then $\alpha(I) = \beta(I)$ but $\mu_\beta(I)$ fails to have maximal rank if and only if: $r = 7$, $m = 3l$ and $3 \leq l \leq 7$; or $r = 8$, $m = 6l$ and $9 \leq l \leq 16$; or $r = 8$, $m = 6l + 1$ and $6 \leq l \leq 13$.*

We give the proof in Subsection IV.iv. Our results of Section IV also explicitly determine the modules in a minimal free resolution of $I(m(p_1 + \cdots + p_r))$ for any m and for any $r \leq 9$ general points of $\mathbf{P}^2$.

For $r > 9$ the question remains open, but in Conjecture I.iii.2 we propose for uniform subschemes that the failures in Corollary I.1 above are the only failures for any r general points. We also provide some evidence for this in Section III, using Campanella-like bounds (viz. Lemma II.6, cf. [Cam]) to verify a number of cases of the conjecture for expectedly good fat point subschemes.

We will use the following notational convention. A divisor on a surface X will be denoted with the typeface $\mathtt{C}$. Its class in the divisor class group $\mathrm{Cl}(X)$ (of divisors modulo linear equivalence) will be denoted C, and the corresponding line bundle in $\mathrm{Pic}(X)$ will be $\mathcal{C}$. In certain special cases, we will also use lower case letters to denote divisor classes, and $\mathcal{O}_X(F)$ to denote the line bundle corresponding to a class F. Finally, in certain instances it will be convenient not to discriminate between a divisor class and its corresponding line bundle, which we may do, for example, by writing $H^i(X, F)$ in place of the strictly correct $H^i(X, \mathcal{O}_X(F))$.

I.i. Previous Work

To put the results of this paper into the context of other recent work, let $I \subset R$ be an ideal (where $R = k[x_0, \ldots, x_n]$ is a polynomial ring), homogeneous with respect to the usual grading (in which each indeterminate x_i has degree 1 and constants have degree 0).

A typical approach to understanding I begins with its Hilbert function (which gives the k-vector space dimension dim I_t of each graded component I_t as a function of the degree t). Next one looks at the number $\nu_t(I)$ of elements of degree t in any minimal set of homogeneous generators; this gives the first module in a minimal free resolution for I. Finally, one considers the successive syzygy modules in a minimal free resolution.

In trying to elucidate principles governing the behavior of these aspects of ideals of R, it is natural to regard R as the homogeneous coordinate ring of the projective space $\mathbf{P}^n$ of dimension n, and to begin with ideals associated to subvarieties or subschemes of $\mathbf{P}^n$. (The reader will recall the usual bijection $X \mapsto I(X)$ from closed subschemes of $\mathbf{P}^n$ to saturated homogeneous ideals of R.)

Points being the geometrically simplest subschemes, one is naturally attracted to studying ideals of the form $I(m_1p_1 + \cdots + m_rp_r)$, for distinct points $p_1, \ldots, p_r \in \mathbf{P}^n$ and nonnegative integers m_i, not all 0, where $I(m_1p_1 + \cdots + m_rp_r)$ denotes the homogeneous ideal generated by all forms which vanish at each point p_i with multiplicity at least m_i. Following Geramita, the corresponding subscheme $m_1p_1 + \cdots + m_rp_r$ is called a *fat point subscheme* and its ideal $I(m_1p_1 + \cdots + m_rp_r)$ is called a *fat point ideal*.

For general points $p_1, \ldots, p_r$, the ideals $I(p_1+\cdots+p_r)$ have been studied extensively (viz., [HS], [Lor] and [EP]). In this situation, the Hilbert function is known trivially (each point imposing independent conditions on forms of each degree until no forms of that degree remain) so attention has focused on numbers of generators and on resolutions. Of particular interest here is the *Ideal Generation Conjecture* (IGC) of [GO] and [GGR]:

Ideal Generation Conjecture I.i.1: *The ideal $I(Z)$ has the maximal rank property for any general set $Z = p_1 + \cdots + p_r$ of r points in $\mathbf{P}^n$.*

To see its relevance, note for any homogeneous ideal $J \subset R$ that $\nu_{t+1}(J)$ is the dimension of the cokernel of the multiplication map $\mu_t(J) : J_t \otimes R_1 \to J_{t+1}$ defined for $f \in J_t$ by $f \otimes x_i \mapsto x_i f$. If the Hilbert function of J is known (and thus the dimensions of $J_t \otimes R_1$ and J_{t+1}), then the rank of $\mu_t(J)$ determines dim cok $\mu_t(J) = \nu_{t+1}(J)$.

Although this conjecture remains open in general, it has been verified in various cases (see [Bl], [GM], [HS], [HSV], [Lor], [O], [Ra]), including $n = 2$ for all r [GGR]. In addition, on $\mathbf{P}^2$ a minimal free resolution of $I = I(Z)$ is of the form $0 \to F_1 \to F_0 \to I \to 0$, where $F_0 = \oplus_t R[-t]^{\nu_t(I)}$. Thus given the number $\nu_t(I)$ of generators for each t and the Hilbert function of I, one knows the Hilbert function of F_1 and hence one knows F_1 itself. In particular, the problem of determining the minimal free resolution of $I(Z)$ on $\mathbf{P}^2$ reduces to determining the Hilbert function and numbers $\nu_t(I)$ of generators, and is thus completely solved for any general set $Z = p_1 + \cdots + p_r \subset \mathbf{P}^2$.

Much less is known or even conjectured in the situation $m_1p_1 + \cdots + m_rp_r$ of fat points, in which the coefficients m_i need not be at most 1. Most work either restricts r, n or the coefficients m_i. For example, [Cat] completely works out the minimal free resolution for any m_i for $r < 6$ general points and $n = 2$ ([Fi] extends this to $r = 6$), while [A], [AH1], [AH2], [AH3], [Hi], [Ch] determine the Hilbert function for any r and n if each m_i is at most 2 and [CM] for any r with $n = 2$ and m_i small and nearly constant. Some steps toward understanding the Hilbert function of generally situated fat points in $\mathbf{P}^n$ have been taken (viz. [I]), but only for $\mathbf{P}^2$ has a conjecture for the Hilbert function of any generally situated finite set of fat points been suggested (first in [H2] and later equivalent variants in [Hi], [Gi] and [H4]).

I.ii. $\mathbf{P}^2$ and its Blowings up

Thus only for $\mathbf{P}^2$ do we have a putative Hilbert function for generally situated fat points, and this begs the questions of what we should expect for the numbers of generators (and hence for the minimal free resolution), given the expected behavior for Hilbert functions. Although it is an open question whether the expected behavior is always obtained, it can in many cases be verified.

We now discuss this in more detail. To do so, we must consider surfaces obtained by blowing up points of $\mathbf{P}^2$. In particular, let $p_1, \ldots, p_r$ be distinct points of $\mathbf{P}^2$. Let $\pi : X \to \mathbf{P}^2$ be the morphism obtained by blowing up each point p_i. Let $\mathtt{E}_i$ denote the exceptional divisor of the blow up of p_i, and let e_i denote its divisor class. Let e_0 denote the pullback to X of the class of a line in $\mathbf{P}^2$; the classes $e_0, \ldots, e_r$ comprise a $\mathbf{Z}$-basis of $\mathrm{Cl}(X)$. Note that this basis, which we call an *exceptional configuration*, is completely determined by π and in turn determines π. Also, recall that $\mathrm{Cl}(X)$ supports an intersection form with respect to which the basis $e_0, \ldots, e_r$ is orthogonal, satisfying $-1 = -e_0^2 = e_1^2 = \cdots = e_r^2$, and that the canonical class K_X of X is $K_X = -3e_0 + e_1 + \cdots + e_r$. Recall that a divisor class is *numerically effective* if its intersection with every effective divisor is nonnegative, and that a prime divisor $\mathtt{C}$ on X with $C^2 = -1 = C \cdot K_X$ is smooth and rational, called a (-1)-*curve*, or an *exceptional curve*. We refer to its class C as a (-1)-*class* or an *exceptional class*. It is known precisely which classes are exceptional classes, when $p_1, \ldots, p_r$ are sufficiently general.

To establish the connection to fat points, consider a fat point subscheme $Z = m_1p_1 + \cdots + m_rp_r \subset \mathbf{P}^2$. Let X be obtained by blowing up each point p_i and let $e_0, \ldots, e_r$ be the corresponding exceptional configuration. Let F_d denote the class $de_0 - m_1e_1 - \cdots - m_re_r$. Since e_0 corresponds to the pullback $\pi^*(\mathcal{O}_{\mathbf{P}^2}(1))$ of the class of a line, we have for each d and i a natural isomorphism of $H^i(X, \mathcal{F}_d)$ with $H^i(\mathbf{P}^2, \pi_*(\mathcal{O}_X(-m_1e_1 - \cdots - m_re_r)) \otimes \mathcal{O}_{\mathbf{P}^2}(d)) = H^i(\mathbf{P}^2, \mathcal{I}_Z(d))$. In particular, the homogeneous coordinate ring $R = \bigoplus_{d\geq 0} H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d))$ can be identified with $\bigoplus_{d\geq 0} H^0(X, de_0)$, and the homogeneous ideal $I(Z) = \bigoplus_{d\geq 0} H^0(\mathbf{P}^2, \mathcal{I}_Z(d))$ in R can be identified with $\bigoplus_{d\geq 0} H^0(X, \mathcal{F}_d)$. Moreover, under these identifications, the natural homomorphisms $H^0(X, \mathcal{F}_d) \otimes H^0(X, e_0) \to H^0(X, \mathcal{F}_{d+1})$ and $I(Z)_d \otimes R_1 \to I(Z)_{d+1}$ correspond, so the dimension ν_{d+1} of the cokernel of the latter is equal to the dimension of the cokernel of the former.

Now, suppose $F = F_d$ is the class of an effective divisor. By taking N to comprise the components of negative self-intersection in the fixed locus of $|F|$, we can write $F = H + N$, where H and N are the classes of effective divisors, H is numerically effective with $h^0(X, \mathcal{F}) = h^0(X, \mathcal{H})$, and N is a sum of prime divisors of negative self-intersection with $h^0(X, \mathcal{N}) = 1$. If the points $p_1, \ldots, p_r$ are general, in all known cases it is true that $h^1(X, \mathcal{H}) = 0$ and that N is a sum of multiples of classes of disjoint exceptional curves disjoint from a general element of $|H|$. In such a case, since the exceptional classes are known, we can explicitly determine $N = -\sum(E \cdot F)E$ (where the sum is over all exceptional classes E with $E \cdot F < 0$), and hence the value $h^0(X, \mathcal{F}) = (H^2 - H \cdot K_X)/2 + 1$ of the Hilbert function of $I(Z)$ in degree d. Assuming the foregoing behavior always holds, we can also explicitly determine whether F_d is the class of an effective divisor (see [H1], [H5]). The point of this paper is to assume the foregoing situation holds, and study the consequences for determining numbers of generators. Toward this end, we make the following definition.

Definition I.ii.1: Let $Z = m_1p_1 + \cdots + m_rp_r \subset \mathbf{P}^2$ be a fat point subscheme, let X be the blowing up of the points p_i and let $F_t = te_0 - m_1e_1 - \cdots - m_re_r$. Then we say Z is *expectedly good* if $F_{\alpha(I(Z))} = H + N$, where H is numerically effective and N is a nonnegative sum of exceptional classes with $h^0(X, \mathcal{O}_X(F_{\alpha(I(Z))})) = h^0(X, \mathcal{H})$, $h^1(X, \mathcal{H}) = 0$ and $h^0(X, \mathcal{N}) = 1$. (It easily follows that $H \cdot N = 0$ and thus that $N = -\sum(E \cdot F)E$, where the sum is over all exceptional classes E with $E \cdot F < 0$.) We also say that the points $p_1, \ldots, p_r \subset \mathbf{P}^2$ are *expectedly good* if the only prime divisors on X of negative self-intersection are exceptional curves and if for every effective and numerically effective divisor $\mathtt{C}$ we have $h^1(X, \mathcal{O}_X(\mathtt{C})) = 0$.

Note that if $p_1, \ldots, p_r$ are expectedly good, then so is any $Z = m_1p_1 + \cdots + m_rp_r$, and, if Z is expectedly good, one only needs to know the classes of exceptional curves and the coefficients m_i in order to compute the Hilbert function of $I(Z)$.

By [H5], $r \leq 8$ general points $p_1, \ldots, p_r \in \mathbf{P}^2$ are expectedly good, and each $Z = m_1p_1 + \cdots + m_9p_9$ is expectedly good for general points $p_1, \ldots, p_9$. Any 9 sufficiently general points, by which we mean the complement of a countable union of closed conditions (which is nonempty unless k is the algebraic closure of a finite field), are also expectedly good. On the other hand, three or more collinear points, or six or more on a conic, or the nine base points of a cubic pencil are not expectedly good. Whether 10 or more sufficiently general points are expectedly good is unknown, but they are expected to be, and conjectures to this effect have been put forward (viz. [H2], [Hi], [Gi] and [H4]). Moreover, many specific examples of expectedly good fat point subschemes $Z = m_1p_1 + \cdots + m_rp_r$ are known with $r > 9$.

I.iii. A Generalized IGC

Let us say that the Uniform Maximal Rank Property (UMRP) on $\mathbf{P}^n$ holds for r if, for each $m > 0$, the maximal rank property for $I(mp_1 + \cdots + mp_r)$ holds for general points $p_1, \ldots, p_r$ of $\mathbf{P}^n$. Let us also say that the Restricted Uniform Maximal Rank Property (RUMRP) on $\mathbf{P}^n$ holds for r if $\mu_{\beta(I(mp_1+\cdots+mp_r))}$ has maximal rank for each $m > 0$ for general points $p_1, \ldots, p_r$ of $\mathbf{P}^n$. We will show in Section IV that:

Theorem I.iii.1: *Let $r \leq 9$. Then the UMRP on $\mathbf{P}^2$ holds if and only if r is 1, 4, or 9, and the RUMRP holds if and only if r is not 7 or 8.*

For general points $p_1, \ldots, p_r \in \mathbf{P}^2$, failures of maximal rank seem for uniform Z to be confined to small r. For example, the failure of the UMRP on $\mathbf{P}^2$ when r is a nonsquare less than 9 is, by Proposition II.4, guaranteed by the existence of uniform abnormal curves for such r. (Following Nagata [N1], a curve $\mathtt{C} \subset \mathbf{P}^2$ of degree d whose multiplicity at each point p_i is at least m_i is said to be *abnormal* if $d\sqrt{r} < m_1 + \cdots + m_r$, and *uniform* if $m_1 = \cdots = m_r$.) But Nagata [N1] proves that no abnormal curves occur for r generic points when r is a square, and he [N2] conjectures that none occur for $r > 9$. Additional evidence that we present in Section III also suggests that the RUMRP may hold on $\mathbf{P}^2$ for $r > 9$. Moreover, for $r > 9$ expectedly good points, RUMRP implies UMRP by Corollary III.6. This prompts us, with some temerity perhaps, to propose a generalized IGC for uniform fat points:

Conjecture I.iii.2: *The UMRP on $\mathbf{P}^2$ holds for all $r > 9$.*

This also suggests the following question:

Question I.iii.3: *Is there an N depending on n, such that the UMRP holds on $\mathbf{P}^n$ for each $r \geq N$?*

II. Background on Surfaces

For the rest of this paper, R will denote the homogeneous coordinate ring $R = k[x_0, x_1, x_2]$ of $\mathbf{P}^2$ (over any algebraically closed field k). Let X be obtained by blowing up distinct points $p_1, \ldots, p_r \in \mathbf{P}^2$ and let $e_0, \ldots, e_r$ be the corresponding exceptional configuration. Let F_d denote the class $de_0 - m_1e_1 - \cdots - m_re_r$ and let $Z = m_1p_1 + \cdots + m_rp_r$.

Following [Mu], we will denote the kernel of $H^0(X, \mathcal{F}_d) \otimes H^0(X, e_0) \to H^0(X, \mathcal{F}_{d+1})$ by $\mathcal{R}(\mathcal{F}_d, e_0)$ and the cokernel by $\mathcal{S}(\mathcal{F}_d, e_0)$; it is then convenient to denote their dimensions by $\mathcal{r}(\mathcal{F}_d, e_0)$ and $\mathcal{s}(\mathcal{F}_d, e_0)$. Note that to say that $I(Z)_d \otimes R_1 \to I(Z)_{d+1}$, or equivalently that $H^0(X, \mathcal{F}_d) \otimes H^0(X, e_0) \to H^0(X, \mathcal{F}_{d+1})$, has maximal rank is just to say that $[\mathcal{R}(\mathcal{F}_d, e_0)][\mathcal{S}(\mathcal{F}_d, e_0)] = 0$.

First we have:

Proposition II.1: *Let T be a closed subscheme of projective space, let $\mathcal{A}$ and $\mathcal{B}$ be coherent sheaves on T and let $\mathcal{C}$ be the class of an effective divisor $\mathtt{C}$ on T.*

(a) *If the restriction homomorphisms $H^0(T, \mathcal{A}) \to H^0(\mathtt{C}, \mathcal{A} \otimes \mathcal{O}_{\mathtt{C}})$ and $H^0(T, \mathcal{A} \otimes \mathcal{B}) \to H^0(C, \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{O}_{\mathtt{C}})$ are surjective (for example, if $h^1(T, \mathcal{A} \otimes \mathcal{C}^{-1}) = 0 = h^1(T, \mathcal{A} \otimes \mathcal{C}^{-1} \otimes \mathcal{B})$), then we have an exact sequence*

$$0 \to \mathcal{R}(\mathcal{A} \otimes \mathcal{C}^{-1}, \mathcal{B}) \to \mathcal{R}(\mathcal{A}, \mathcal{B}) \to \mathcal{R}(\mathcal{A} \otimes \mathcal{O}_{\mathtt{C}}, \mathcal{B}) \to$$
$$\mathcal{S}(\mathcal{A} \otimes \mathcal{C}^{-1}, \mathcal{B}) \to \mathcal{S}(\mathcal{A}, \mathcal{B}) \to \mathcal{S}(\mathcal{A} \otimes \mathcal{O}_{\mathtt{C}}, \mathcal{B}) \to 0.$$

(b) *If $H^0(T, \mathcal{B}) \to H^0(C, \mathcal{B} \otimes \mathcal{O}_{\mathtt{C}})$ is surjective (for example, if $h^1(T, \mathcal{B} \otimes \mathcal{C}^{-1}) = 0$), then $\mathcal{S}(\mathcal{A} \otimes \mathcal{O}_{\mathtt{C}}, \mathcal{B}) = \mathcal{S}(\mathcal{A} \otimes \mathcal{O}_{\mathtt{C}}, \mathcal{B} \otimes \mathcal{O}_{\mathtt{C}})$.*

(c) *If T is a smooth curve of genus g, and $\mathcal{A}$ and $\mathcal{B}$ are line bundles of degrees at least $2g+1$ and $2g$, respectively, then $\mathcal{S}(\mathcal{A}, \mathcal{B}) = 0$.*

Proof: See [Mu] for (a) and (c); we leave (b) as an easy exercise for the reader. ◇

Let F be the class of an effective divisor $\mathtt{F}$ on a surface X. Let $F = H + N$ be a Zariski decomposition of F (i.e., $h^0(X, \mathcal{F}) = h^0(X, \mathcal{H})$ and $h^0(X, \mathcal{N}) = 1$; for example, N could be the class of the fixed part of the linear system $|\mathtt{F}|$ and then $H = F - N$ would be the free part of F). The following lemma allows us to reduce a consideration of $\mathcal{S}(\mathcal{F}, e_0)$ to one of $\mathcal{S}(\mathcal{H}, e_0)$.

Lemma II.2: *Let $e_0, \ldots, e_r$ be the exceptional configuration corresponding to a blowing up $\pi : X \to \mathbf{P}^2$ at distinct points $p_1, \ldots, p_r$, and let F be a divisor class on X. If F is not the class of an effective divisor, then $s(\mathcal{F}, e_0) = h^0(X, F + e_0)$. If F is the class of an effective divisor, let $F = H + N$ be a Zariski decomposition; then $s(\mathcal{F}, e_0) = [h^0(X, F + e_0) - h^0(X, H + e_0)] + s(\mathcal{H}, e_0)$.*

Proof: See Lemma 2.10 of [H7]. ◇

Remark II.3: To determine $\nu_t(I(Z))$ for each t for some fat point subscheme $Z = m_1p_1 + \cdots + m_rp_r$ of $\mathbf{P}^2$, by Lemma II.2 it is enough on the blow up X of $\mathbf{P}^2$ at $p_1, \ldots, p_r$ to determine $h^0(X, de_0 - m_1e_1 - \cdots - m_re_r)$ for every d, and, for each d such that $h^0(X, de_0 - m_1e_1 - \cdots - m_re_r) > 0$, to determine: the free part H of $de_0 - m_1e_1 - \cdots - m_re_r$; $s(\mathcal{H}, e_0)$; and $h^0(X, H + e_0)$. (It is not hard to see that being able to compute $h^0(X, \mathcal{F})$ for any F also lets one find the free part of any F whenever F is the class of an effective divisor. And once one knows $\nu_t(I(Z))$ for all t, one also knows the modules in a minimal free resolution $0 \to F_1 \to F_0 \to I(Z) \to 0$ of $I(Z)$; viz. Example IV.ii.2.)

In the case of any $r \leq 9$ points, the results of [H6] provide a solution to determining $h^0(X, \mathcal{F})$ for any class F, and thus to finding the free part of F when $h^0(X, \mathcal{F}) > 0$. For $r \leq 9$ general points, these results are well known and can, in any case, be recovered from [H6] or [H1]; for the reader's convenience we recall the facts relevant to a uniform class F in the case of r general points of $\mathbf{P}^2$, first for $r \leq 8$, and then for $r = 9$. (A class F on a blowing up X of $\mathbf{P}^2$ at distinct points $p_1, \ldots, p_r$ will be called a *uniform* class if $F = de_0 - m(e_1 + \cdots + e_r)$ for some nonnegative integers d and m.)

Let X be the blowing up of $r \leq 8$ general points of $\mathbf{P}^2$. If F is uniform and if it is the class of an effective divisor, then the fixed part N is also uniform, equal to $-\sum(E \cdot F)E$, where the sum is over all classes E of (-1)-curves with $E \cdot F < 0$. The classes of the (-1)-curves are known; up to permutation of the indices, they are (see Section 26 of [Ma]): e_1, $e_0 - e_1 - e_2$, $2e_0 - (e_1 + \cdots + e_5)$, $3e_0 - (2e_1 + e_2 + \cdots + e_7)$, $4e_0 - (2e_1 + 2e_2 + 2e_3 + e_4 + \cdots + e_8)$, $5e_0 - (2e_1 + \cdots + 2e_6 + e_7 + e_8)$, and $6e_0 - (3e_1 + 2e_2 + \cdots + 2e_8)$. Now one can show that $N = 0$ if $r = 1$ or 4; otherwise, N is a nonnegative multiple of: $e_0 - e_1 - e_2$ if $r = 2$; $3e_0 - 2e_1 - 2e_2 - 2e_3$ if $r = 3$; $2e_0 - (e_1 + \cdots + e_5)$ for $r = 5$; $12e_0 - 5(e_1 + \cdots + e_6)$, $r = 6$; $21e_0 - 8(e_1 + \cdots + e_7)$, $r = 7$; or $48e_0 - 17(e_1 + \cdots + e_8)$, $r = 8$. It also follows that a uniform class $de_0 - m(e_1 + \cdots + e_r)$ is the class of an effective divisor if and only if $d \geq \epsilon_r m$, where $\epsilon_1 = \epsilon_2 = 1$, $\epsilon_3 = 3/2$, $\epsilon_4 = \epsilon_5 = 2$, $\epsilon_6 = 12/5$, $\epsilon_7 = 21/8$, and $\epsilon_8 = 48/17$.

Now, the free part of the class of an effective divisor is numerically effective, and, if X is any blowing up of $\mathbf{P}^2$ at 8 or fewer points, general or not, then ([H5], [H6]) any numerically effective class F on X is the class of an effective divisor with no fixed components and has $h^1(X,\mathcal{F}) = h^2(X,\mathcal{F}) = 0$, hence $h^0(X,\mathcal{F}) = (F^2 - K_X\cdot F)/2+1$ by Riemann–Roch for surfaces.

Finally, let $r=9$. Nine general points of $\mathbf{P}^2$ always lie on a smooth cubic curve, so more generally let X be the blowing up of any $r=9$ distinct points of a smooth cubic curve $\mathtt{C}'$ on $\mathbf{P}^2$. Then $-K_X = 3e_0-e_1-\cdots-e_9$ is numerically effective, the class of a smooth elliptic curve $\mathtt{C}$, the proper transform of $\mathtt{C}'$. If F is uniform, we can write $F = te_0 - sK_X$ for uniquely determined integers t and s, with $s\geq 0$. Moreover, F is the class of an effective divisor if and only if t is also nonnegative, in which case $h^2(X,\mathcal{F}) = 0$, hence $h^0(X,\mathcal{F}) = (F^2-K_X\cdot F)/2+1+h^1(X,\mathcal{F})$. In addition, if $t>0$, then F is fixed part free and $h^1(X,\mathcal{F}) = 0$. If, however, $t=0$, things are more delicate. If the restriction of $\mathcal{O}_X(-K_X)$ to $\mathtt{C}$ has infinite order in Pic$(\mathtt{C})$, let $a=0$. Otherwise, let l be the order of the restriction of $\mathcal{O}_X(-K_X)$ to $\mathtt{C}$, and define a and b via $s = al+b$ where $0\leq b<l$. Then $h^0(X,-sK_X) = a+1$ and $h^1(X,-sK_X) = a$. (Note for an algebraically closed field k which is not the algebraic closure of a finite field, that for sufficiently general—i.e., a nonempty complement of a countable union of closed conditions—points $p_1,\dots,p_9$ no nonzero power of $\mathcal{O}_X(-K_X)$ restricts trivially to $\mathtt{C}$. If k is the algebraic closure of a finite field, however, then the restriction of $\mathcal{O}_X(-K_X)$ to $\mathtt{C}$ always has finite order.)

The next result will be helpful in verifying failure of the UMRP. Call a uniform class $E = de_0 - m(e_1+\cdots+e_r)$ on a blowing up X of $\mathbf{P}^2$ at distinct points $p_1,\dots,p_r$ *abnormal* if E is the class of an effective divisor with $d<\sqrt{r}m$ (note that this is equivalent to $E^2<0$).

Proposition II.4: *Let X be a blowing up of r distinct points $p_1,\dots,p_r$ of $\mathbf{P}^2$. If X has a uniform abnormal class E, then for some positive integers n and m, $I(m(p_1+\cdots+p_r))_n\otimes R_1\to I(m(p_1+\cdots+p_r))_{n+1}$ does not have maximal rank.*

Proof: Since E is the class of an effective divisor of negative self-intersection, we can find positive integers a and b such that ae_0+bE has nontrivial fixed part but such that $(a+1)e_0+bE$ has trivial fixed part. Now, $ae_0+bE = ne_0-m(e_1+\cdots+e_r)$ for some positive n and m. Since $a>0$, $H^0(X,ae_0)\otimes H^0(X,e_0)\to H^0(X,(a+1)e_0)$ is not injective, hence neither is $H^0(X,ae_0+bE)\otimes H^0(X,e_0)\to H^0(X,(a+1)e_0+bE)$. Since $(a+1)e_0+bE$ is fixed component free but ae_0+bE is not, we see $H^0(X,ae_0+bE)\otimes H^0(X,e_0)\to H^0(X,(a+1)e_0+bE)$ is also not surjective. Thus $H^0(X,ae_0+bE)\otimes H^0(X,e_0)\to H^0(X,(a+1)e_0+bE)$, and hence $I(m(p_1+\cdots+p_r))_n\otimes R_1\to I(m(p_1+\cdots+p_r))_{n+1}$, do not have maximal rank. $\diamondsuit$

The following result is well known (see Proposition 3.7 of [DGM]) and follows easily by appropriately applying Proposition II.1 (or by noting that τ_Z+1 is just the regularity $\sigma(I(Z))$ of $I(Z)$).

Lemma II.5: *Let $e_0,\dots,e_r$ be the classes corresponding to a blowing up $X\to\mathbf{P}^2$ at distinct points $p_1,\dots,p_r$. Let $Z = m_1p_1+\cdots+m_rp_r$, and let F_d denote $de_0-m_1e_1-\cdots-m_re_r$. If ω_Z is the degree of a generator of greatest degree in a minimal set of homogeneous generators of $I(Z)$ (equivalently, μ_d is surjective for $d\geq\omega_Z$ but not for $d=\omega_Z-1$) and if τ_Z is the least integer t such that $h^1(X,\mathcal{F}_t)=0$, then $\omega_Z\leq\tau_Z+1$. In particular, $\mathcal{S}(\mathcal{F}_t,e_0)=0$ for $t>\tau_Z$.*

We now determine Campanella-like bounds (cf. [Cam]). Let $F = a_0e_0-a_1e_1-\cdots-a_re_r$, with $a_i>0$ for all i, be the class of an effective divisor on a blow up X of $\mathbf{P}^2$ at distinct points $p_1,\dots,p_r\in\mathbf{P}^2$. Let $h = h^0(X,\mathcal{F})$, $l_i = h^0(X,F-(e_0-e_i))$, and $q_i = h^0(X,F-e_i)$.

Lemma II.6: *Given the multiplication map $\mu: H^0(X,\mathcal{F})\otimes H^0(X,e_0)\to H^0(X,F+e_0)$ and $0<i\leq r$, we have:*

$$\max(l_i, 3h-h^0(X,F+e_0))\leq \textit{dim ker}\ \mu\leq l_i+q_i.$$

Proof: For specificity, take $i=1$. Let x (y and z, resp.) be the equation of the line through p_2 and p_3 (resp., p_1 and p_3, and p_1 and p_2). Let L be the image of $\Gamma(e_0-e_1)$ in $\Gamma(e_0)$, where Γ is the global sections functor. Thus L can be regarded as the vector space span of y and z, making $z\Gamma(\mathcal{F})+y\Gamma(\mathcal{F})$ the image of $\Gamma(\mathcal{F})\otimes L$ under μ. It has dimension $2h-l_1$ since $z\Gamma(\mathcal{F})\cap y\Gamma(\mathcal{F}) = zy\Gamma(F-(e_0-e_1))$, where we regard the intersection as taking place in $\Gamma(((F \cdot e_0) + 1)e_0)$. Therefore, $l_1 \leq \dim \ker \mu$. But since $\Gamma(\mathcal{F}) \otimes \Gamma(e_0)$ has dimension $3h$ and μ maps into $H^0(X, F + e_0)$, it is clear that we also have $3h - h^0(X, F + e_0) \leq \dim \ker \mu$ and hence $\max(l_1, 3h - h^0(X, F + e_0)) \leq \dim \ker \mu$.

To bound dim ker μ above, note that all elements of $z\Gamma(\mathcal{F}) + y\Gamma(\mathcal{F})$ correspond to forms on $\mathbf{P}^2$ that vanish at p_1 to order at least $a_1 + 1$. Thus $(y\Gamma(\mathcal{F}) + z\Gamma(\mathcal{F})) \cap x\Gamma(\mathcal{F})$ lies in the image of $x\Gamma(\mathcal{F} - e_1)$ under the natural inclusion $x\Gamma(F - e_1) \subset x\Gamma(\mathcal{F})$, so $\dim \mathrm{Im}\ \mu \geq (2h - l_1) + (h - q_1)$ hence $\dim \ker \mu \leq l_1 + q_1$. ◊

Corollary II.7: *Let F and μ be as in Lemma II.6, let $d = F \cdot e_0$ and assume $h^1(X, F) = 0$.*

(a) Then μ has maximal rank if and only if $\max(0, 2h - d - 2) = \dim \ker \mu$.

(b) Moreover, $\max(0, 2h - d - 2) \leq \dim \ker \mu \leq l_1 + q_1$.

(c) If $h^1(X, F - (e_0 - e_1)) = h^1(X, F - e_1) = 0$, then $l_1 + q_1 = 2h - d - 2$.

Proof: We use the notation of Lemma II.6.

(a) Clearly, μ has maximal rank if and only if $\max(0, 3h - h^0(X, F + e_0)) = \dim \ker \mu$. But $h^1(X, F) = 0$ (and hence $h^1(X, F + e_0) = 0$), so by Riemann–Roch we compute $h^0(X, F + e_0) = h + d + 2$. Thus $3h^0(X, F) - h^0(X, F + e_0) = 2h - d - 2$ and the result follows.

(b) This follows by the proof of (a) and by Lemma II.6.

(c) Let $m = F \cdot e_1$. Since $h^1(X, F - e_1) = 0$, taking $\mathtt{E}$ to be the effective divisor whose class is e_1, the exact sheaf sequence $0 \to \mathcal{O}_X(F - e_1) \to \mathcal{F} \to \mathcal{O}_{\mathtt{E}} \otimes \mathcal{F} \to 0$ is exact on global sections, so $h = h^0(X, F) = h^0(X, F - e_1) + h^0(\mathtt{E}, \mathcal{O}_X(F) \otimes \mathcal{O}_{\mathtt{E}}) = q_1 + m + 1$.

Since $h^1(X, F - (e_0 - e_1)) = 0$, taking $\mathtt{C}$ to be a general effective divisor whose class is $e_0 - e_1$, the exact sheaf sequence $0 \to \mathcal{O}_X(F - (e_0 - e_1)) \to \mathcal{F} \to \mathcal{O}_{\mathtt{C}} \otimes \mathcal{F} \to 0$ is exact on global sections. Computing dimensions we find $h = l_1 + d + 1 - m$ so $2h - d - 2 = l_1 + (h - m - 1) = l_1 + q_1$. ◊

III. Applying the Bounds

Let X be the blow up of $\mathbf{P}^2$ at distinct points $p_1, \ldots, p_r \in \mathbf{P}^2$. Let $e_0, \ldots, e_r$ be the corresponding exceptional configuration, and define the *roots* $\rho_0 = e_0 - e_1 - e_2 - e_3$, $\rho_i = e_i - e_{i+1}$, $i > 0$. Reflections $s_i(x) = x + (x \cdot \rho_i)\rho_i$ through each ρ_i define intersection form-preserving involutions of $\mathrm{Cl}(X)$, generating a subgroup W (infinite for $r > 8$), called the *Weyl* group, of the orthogonal group on $\mathrm{Cl}(X)$. Let us say that $p_1, \ldots, p_r$ are *strongly nonspecial* if $h^0(X, \mathcal{F}) = h^0(X, \mathcal{O}_X(wF))$ for all $w \in W$ and $F \in \mathrm{Cl}(X)$. This is somewhat stronger than but implies Nagata's condition of being nonspecial for Cremona transformations [N1]. And just as points which are independent generic points over the prime field are nonspecial for Cremona transformations [N1], they are also strongly nonspecial. Nor is it hard to check that expectedly good points are strongly nonspecial. As a further example, over any algebraically closed ground field k, sufficiently general smooth points of a cuspidal cubic $\mathtt{C}'$ are strongly nonspecial. (By sufficiently general, taking X to be the blow up of $\mathbf{P}^2$ at the points and $\mathtt{C}$ to be the proper transform to X of $\mathtt{C}'$, we mean such that the kernel of the induced homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(\mathtt{C})$ is trivial in characteristic 0 or is $pK_X^{\perp}$ in characteristic p, where $K_X^{\perp}$ is the subgroup of elements $\mathcal{F}$ with $F \cdot K_X = 0$. For justification, see Example 3.4 of [H3], and use [H1].)

We will obtain some asymptotic results that essentially say that some property holds for all but finitely many elements of a Weyl group orbit. The next lemma determines some properties of these orbits, including that they tend to be infinite.

Lemma III.1: *Let $F \neq 0$ be a numerically effective class on the blowing up X of strongly nonspecial points $p_1, \ldots, p_r \in \mathbf{P}^2$, where $e_0, \ldots, e_r$ is the corresponding exceptional configuration.*

(a) The orbit WF under the Weyl group action is infinite if and only if $r > 9$, or $r = 9$ but $F \neq -lK_X$ for any $l > 0$.

(b) The class $wF - e_0$ is the class of an effective divisor for at most finitely many elements wF of WF.

Proof: (a) The forward implication is clear since W is finite for $r < 9$, and for $r = 9$, W stabilizes $-K_X$, so assume $r > 9$, or $r = 9$ but $F \neq -lK_X$. Since $p_1, \ldots, p_r$ are strongly nonspecial, if H is the class of an effective divisor, so is wH, for every $w \in W$. Thus $wF \cdot H = F \cdot (w^{-1}H) \geq 0$, whenever H is the class of an effective divisor; i.e., wF is numerically effective for every $w \in W$. Now, $F^2 \geq 0$ (see, e.g., Proposition 4 of [H5]); we will first consider the case that $F^2 > 0$. Then, by the index theorem, the subgroup $F^\perp \subset \mathrm{Cl}(X)$ orthogonal to F is negative definite, so the stabilizer of F in W is finite. Therefore, WF is infinite if W is, which it is for $r \geq 9$.

Now suppose $F^2 = 0$. Since $e_0 \cdot wF \geq 0$ for every $w \in W$, there is a particular w such that $e_0 \cdot wF$ is minimal. Let us write $wF = b_0e_0 - b_1e_1 - \cdots - b_re_r$ for some integers b_i. Reflections through the roots ρ_i, $i > 0$, just permute the coefficients $b_1, \ldots, b_r$, so we may assume that $b_1 \geq b_2 \geq \cdots \geq b_r$. In this case, if $\rho_0 \cdot wF < 0$, then $s_0wF \cdot e_0 < wF \cdot e_0$, contrary to assumption, so we have $wF \cdot \rho_i \geq 0$ for every $i \geq 0$. It is not hard to show that this implies that wF is a nonnegative integer linear combination of the classes $H_0 = e_0$, $H_1 = e_0 - e_1$, $H_2 = 2e_0 - e_1 - e_2$, $H_i = 3e_0 - e_1 - \cdots - e_i$, $2 < i \leq r$; i.e., $wF = \sum_i h_iH_i$ with $h_i \geq 0$. If $h_i > 0$ for some $i > 9$, let $D = \sum_{i\leq 9} h_iH_i + \sum_{i>9} h_iH_9$. Then $D^2 > 0$ and D is the sum of wF and nonnegative multiples of $e_{10}, \ldots$, all of which are orthogonal to D, so D is numerically effective. Thus by a previous case $W'D$ is infinite, where W' is the subgroup of W generated by $s_0, \ldots, s_8$. But W' stabilizes $e_{10}, \ldots$, so also $W'F$ and thus WF are infinite.

So suppose that $h_i = 0$ for all $i > 9$. Then using $0 \neq wF = \sum_{i\leq 9} h_iH_i$ and $F^2 = 0$, it is easy to check that either $wF = h_1H_1$ or $wF = h_9H_9$. Since $H_9 = -K_X$, if $wF = h_9H_9$, then $r > 9$ by hypothesis. Let W' now denote the subgroup generated by $s_0, \ldots, s_9$; it suffices to show $W'F$ is infinite. I.e., it suffices to consider the case $r = 10$. But if $r = 10$, then $H_9 = -K_X + e_{10}$. As is well known, W fixes K_X while We_{10} is infinite (indeed, We_{10} is the set of all classes of (-1)-curves on X), so WH_9 must also be infinite.

Finally we check that WH_1 is infinite. First, $\rho = 2e_0 - e_4 - \cdots - e_9$ is in $W\rho_1$, so reflection s by ρ is in W, and it is easy to check explicitly that the composition s_0s generates a cyclic subgroup W'' of W such that $W''H_1$ is infinite.

(b) If $wF - e_0$ is the class of an effective divisor, then numerical effectivity of wF implies that $wF \cdot (wF - e_0) \geq 0$. Thus it suffices to show $wF \cdot (wF - e_0) < 0$, or equivalently $F^2 < wF \cdot e_0$, for all but finitely many $wF \in WF$. In fact, for any integer N it is true that $N < e_0 \cdot wF$, for all but finitely many $wF \in WF$. For suppose for each D in an infinite subset $V \subset WF$ we had $e_0 \cdot D \leq N$. Then, writing each D as $D = b_0e_0 - b_1e_1 - \cdots - b_re_r$ for integers b_i depending on D, we would have infinitely many integer solutions $b_0, \ldots, b_r$ to $F^2 = b_0^2 - b_1^2 - \cdots - b_r^2$ with $0 \leq b_0 = e_0 \cdot D \leq N$, which is impossible. ♢

The next result applies Lemma II.6 to give a maximal rank criterion.

Lemma III.2: *With X as in Lemma III.1, let G be the class of an effective, numerically effective divisor. If $w \in W$ is such that there exists an $i > 0$ with $G^2 < e_i \cdot w(G)$ and $G^2 < (e_0 - e_i) \cdot w(G)$, then $\mu : \Gamma(w\mathcal{G}) \otimes \Gamma(e_0) \to \Gamma(e_0 + G)$ is injective, and so has maximal rank.*

Proof: By $G^2 < e_i \cdot w(G)$ we have $(w(G))^2 < e_i \cdot w(G)$, but, since w preserves the monoid of classes of effective divisors, wG is numerically effective, so $wG - e_i$ is not the class of an effective divisor; thus $q_i = h^0(X, wG - e_i) = 0$. Similarly, $(wG - (e_0 - e_i)) \cdot w(G) < 0$ implies $l_i = h^0(X, wG - (e_0 - e_i)) = 0$. Hence Lemma II.6 implies $\ker \mu = 0$. ♢

We now obtain an asymptotic result. (Given a numerically effective class G on X, Z_G will denote $Z_G = m_1p_1 + \cdots + m_rp_r$, where $m_i = e_i \cdot G$.)

Theorem III.3: *With X as in Lemma III.1, let G be the class of an effective, numerically effective divisor such that $h^1(X, \mathcal{G}) = 0$. Then, for each $w \in W$, $I(Z_{wG})$ has the maximal rank property for all but finitely many elements of $\{Z_{wG} | w \in W\}$.*

Proof: Since G is the class of an effective divisor, so is wG for every $w \in W$, but, for all but finitely many $wG \in WG$, $wG - e_0$ is not, by Lemma III.1. Thus $\alpha(I(Z_{wG})) = wG \cdot e_0$ for all but finitely many $wG \in WG$. On the other hand, $h^1(X, \mathcal{G}) = 0$ (and hence $h^1(X, wG) = 0$), so, for all but finitely many $wG \in WG$, the regularity of $I(Z_{wG})$ is at most $\alpha(I(Z_{wG})) + 1$. Therefore, $\mu_t(I(Z_{wG}))$ has maximal rank except possibly for $t = \alpha(I(Z_{wG}))$; since $\mu_{\alpha(I(Z_{wG}))}$ has maximal rank if and only if $\mu : \Gamma(w\mathcal{G}) \otimes \Gamma(e_0) \to \Gamma(e_0 + wG)$ does, we turn our attention to the latter.

There are clearly only finitely many integer solutions $d, b_1, \cdots, b_r$ to $G^2 = d^2 - b_1^2 - \cdots - b_r^2$ with $\{b_i : 0 < i\}$ bounded. Thus the number of elements in the orbit WG with $\max_{0<i}(wG \cdot e_i) \leq G^2$ is finite.

Thus it is enough by Lemma III.2 to show for each i that $wG\cdot(e_0-e_i)>G^2$ occurs for all but finitely many $wG\in WG$.

We fix $i>0$; then there are only finitely many integer solutions $d,b_1,\cdots,b_r$ to $G^2=d^2-b_1^2-\cdots-b_r^2$ with $\{b_j : 0<j\neq i\}$ bounded (because then $d^2-b_i^2$ takes on only a finite set of values, which factor only a finite number of ways). Thus for all but finitely many $wG\in WG$ we can choose $0<j_w\neq i$ such that $wG\cdot e_{j_w}>G^2$. Now write e_0-e_i as $(e_0-e_i-e_{j_w})+e_{j_w}$. Thus $wG\cdot(e_0-e_i)=wG\cdot((e_0-e_{j_w}-e_i)+e_{j_w})\geq wG\cdot e_{j_w}>G^2$ holds for all but finitely many $wG\in WG$. ♢

To apply Theorem III.3, one needs examples of classes G of numerically effective, effective, and regular (i.e., $h^1=0$) divisors on a blowing up X of $\mathbf{P}^2$ at strongly nonspecial points. It is easy to give examples: Given such an X, if $m_i\geq 0$, then for d sufficiently large (say $d>\sum_i m_i$), $G=de_0-m_1e_1-\cdots-m_re_r$ is such a class.

Alternatively, let X be the blowing up of points $p_1,\ldots,p_r$ which are independent generic over the prime field. If $-K_X\cdot G\geq 0$, then G is effective, numerically effective, and regular if and only if G is in the W-orbit of the nonnegative subsemigroup S of $\mathrm{Cl}(X)$ generated by $\{H_0=e_0, H_1=e_0-e_1, H_2=2e_0-e_1-e_2, H_3=3e_0-e_1-e_2-e_3, H_4=3e_0-e_1-e_2-e_3-e_4,\ldots\}$. The proof is to specialize $p_1,\ldots,p_r$ to a cubic, then use semicontinuity and results of [H1] (also see [H6]).

When $r\geq 9$, W has a particularly tractable subgroup for which a more explicit result analogous to Theorem III.3 can be stated (when $r<9$, W is finite and hence Theorem III.3 is trivial). So assume that $p_1,\ldots,p_r\in\mathbf{P}^2$ are independent generic over the prime field with $r\geq 9$. Let T be the subgroup of $\mathrm{Cl}(X)$ generated by the roots $\rho_1,\ldots,\rho_8$. Then, given any $v\in T$, it turns out that $v\mapsto\tau_v$ defines an injective homomorphism $T\to W$, where we define τ_v via $\tau_v(G)=G+(G\cdot H_9)v-(1/2)(2G\cdot v+(G\cdot H_9)v^2)H_9$. If G is in S with $-K_X\cdot G>0$, then as above G is effective, numerically effective and regular, so, as the proof of Theorem III.3 shows, $I(Z_{\tau_v(G)})$ has the maximal rank property for each $v\in T$ such that $G^2<e_0\cdot\tau_v(G)$, $G^2<e_1\cdot\tau_v(G)$ and $G^2<(e_0-e_1)\cdot\tau_v(G)$. But T is negative definite and $G\cdot H_9\geq -G\cdot K_X>0$, so substituting our expression for $\tau_v(G)$ into $e_0\cdot\tau_v(G)$, $e_1\cdot\tau_v(G)$ and $(e_0-e_1)\cdot\tau_v(G)$, we see $G^2<e_0\cdot\tau_v(G)$, $G^2<e_1\cdot\tau_v(G)$ and $G^2<(e_0-e_1)\cdot\tau_v(G)$ hold for all but finitely many $v\in T$. (In fact, we can be explicit here: these conditions and therefore the maximal rank property for $I(Z_{\tau_v(G)})$ hold if $\sqrt{-v^2}>2+\sqrt{24(G\cdot e_0)/(G\cdot H_9)+2G^2/(G\cdot H_9)}$.)

Although the foregoing paragraph provides a fairly easy method of generating examples $Z=m_1p_1+\cdots+m_rp_r$ for which $I(Z)$ has the maximal rank property, it is also nice to have an explicit criterion in terms of the coefficients m_i for the maximal rank property to hold. We give such a criterion when $r=9$ in the next example.

Example III.4: Let $m_1\geq\cdots\geq m_9\geq 0$. Here we show for general points p_i that $I(Z)$ has the maximal rank property for $Z=m_1p_1+\cdots+m_9p_9$, if $m_1=m_9$ or if $m_9\geq 20(m_1-m_9+1)^2$ and $m_1+\cdots+m_9\not\equiv 2 \pmod 3$.

If $m_1=m_9$, which is to say that Z is uniform, then it follows from Corollary IV.iii.2 that $I(Z)$ has the maximal rank property, so assume that Z is not uniform. Let X be the blowing up of the points p_i and let $e_0,e_1,\ldots,e_9$ be the corresponding exceptional configuration. Since X is obtained by blowing up 9 general points, $-K_X$ is numerically effective, so $-F_{\alpha(Z)}\cdot K_X\geq 0$, where $F_t(Z)=te_0-m_1e_1-\cdots-m_9e_9$ (and we write just F_t when the Z being referred to is unambiguous). Since F_t is not uniform, we have in fact that $-F_{\alpha(Z)}\cdot K_X>0$. Thus $\alpha(Z)\geq d$, where d is the largest integer which is at most $1+(m_1+\cdots+m_9)/3$. Moreover, $F_d^2>0$, so by Riemann–Roch F_d is the class of an effective divisor, hence actually $\alpha(Z)=d$. (To see $F_d^2>0$, let $H=(d-3m_9)e_0-(m_1-m_9)e_1-\cdots-(m_8-m_9)e_8$, so $F_d=H-m_9K_X$. Then $F_d^2=H^2-2m_9H\cdot K_X\geq 2m_9-8(m_1-m_9)^2$, but by assumption we have $m_9\geq 20(m_1-m_9+1)^2$. Note that by the same reasoning $H-lK_X$ is the class of an effective divisor whenever $l\geq 4(m_1-m_9)^2$.)

Now we check that $\alpha(Z)=\beta(Z)$; i.e., that F_d is numerically effective. As we noted above, $H-lK_X$ is effective for $l=4(m_1-m_9)^2$. Because the points p_i are general, the only curves which could occur as fixed components of $|H-lK_X|$ of negative self-intersection are (-1)-curves, and if $\mathtt{E}$ is such a component, then $E\cdot(H-lK_X)<0$. In particular, E is not e_i for any i, so $E\cdot e_0>0$, hence $\mathtt{E}$ occurs with multiplicity at most $e_0\cdot(H-lK_X)=d-3(m_9-l)$. Therefore, since $-E\cdot K_X=1$, we will have $E\cdot(H-tK_X)\geq 0$ if $t-l\geq d-3m_9+3l$, but $d-3m_9\leq 1+((m_1-m_9)+\cdots+(m_8-m_9))/3\leq 4(m_1-m_9)^2$ so $m_9-l\geq 20(m_1-m_9+1)^2-4(m_1-m_9)^2\geq 16(m_1-m_9)^2\geq(d-3m_9)+3l$, so $F_d=H-m_9K_X$ meets every such E nonnegatively, so F_d is numerically effective and $h^1(X,F_d)=0$ by [H6].

By definition of d, it is easy to check $1 \leq -F_d \cdot K_X \leq 3$. Suppose $1 = -F_d \cdot K_X$. Then, keeping in mind that $-K_X$ is numerically effective, $-K_X \cdot (F_d - e_1) = 0$, so $F_d - e_1$, not being uniform but being in $K_X^{\perp}$, cannot be the class of an effective divisor. Likewise, $-K_X \cdot (F_d - (e_0 - e_1)) < 0$, so $F_d - (e_0 - e_1)$ also cannot be the class of an effective divisor. Thus, as in the proofs of Lemma III.2 and Theorem III.3, $I(Z)$ has the maximal rank property.

Now consider the case $3 = -F_d \cdot K_X$. It is easy to check that $F_d(Z) - e_1 = F_d(Z')$, where $Z' = (m_1 + 1)p_1 + m_2p_2 + \cdots + m_9p_9$, and that $F_d(Z) - (e_0 - e_1) = F_{d-1}(Z'')$, where $Z'' = (m_1 - 1)p_1 + m_2p_2 + \cdots + m_9p_9$. Reasoning as above shows that $F_d(Z')$ and $F_{d-1}(Z'')$ are classes of effective, numerically effective divisors meeting $-K_X$ positively (and hence $h^1(X, F_d(Z')) = 0 = h^1(X, F_{d-1}(Z''))$ by [H6]) when $m_9 \geq 20(m_1 - m_9 + 1)^2$. Arguing as above and using Corollary II.7(c) now shows that $I(Z)$ has the maximal rank property.

To view Theorem III.3 from a different perspective, given any distinct points $p_1, \ldots, p_r \in \mathbf{P}^2$, we define an equivalence relation on the set of all fat point subschemes $m_1p_1 + \cdots + m_rp_r$: we say $Z = \sum_i m_ip_i$ and $Z' = \sum_i m'_ip_i$ are *Cremona equivalent* if, with respect to the usual exceptional configuration $e_0, \ldots, e_r$ on the blow up X of $\mathbf{P}^2$ at the points p_i, we have $w(\beta(I(Z))e_0 - m_1e_1 - \cdots - m_re_r) = \beta(I(Z'))e_0 - m'_1e_1 - \cdots - m'_re_r$ for some $w \in W$.

When $p_1, \ldots, p_r$ are strongly nonspecial, what it means for fat point subschemes $Z = \sum_i m_ip_i$ and $Z' = \sum_i m'_ip_i$ to be Cremona equivalent is that their associated linear systems in their respective degrees β are in some sense geometrically the same (regarded as complete linear systems on the blown up surface). Much else can be different, but by the next result most (i.e., all but finitely many) of the representatives in the equivalence class of an expectedly good fat point subscheme will have ideals with the maximal rank property.

Corollary III.5: *Let $p_1, \ldots, p_r \in \mathbf{P}^2$ be strongly nonspecial and let $Z = \sum_i m_ip_i$ be expectedly good. Then for all but finitely many fat point subschemes $Z' = \sum_i m'_ip_i$ Cremona equivalent to Z, $\alpha(I(Z')) = \beta(I(Z'))$ and $I(Z')$ has the maximal rank property.*

Proof: Let X and $e_0, \ldots, e_r$ be as usual and let $G = \beta(I(Z))e_0 - m_1e_1 - \cdots - m_re_r$. By definition, every Z' Cremona equivalent to Z is of the form Z_{wG} for some $w \in W$. However, not every Z_{wG} need be Cremona equivalent to Z, since $\beta(I(Z_{wG}))$ could be less than $wG \cdot e_0$; in fact, from the definition we see that Z_{wG} and Z are Cremona equivalent exactly when $\beta(I(Z_{wG})) = wG \cdot e_0$. We will check that this occurs for all but finitely many of $\{Z_{wG} | w \in W\}$.

First, we have $\alpha(I(Z_{wG})) \leq \beta(I(Z_{wG})) \leq wG \cdot e_0$. By Lemma III.1(b), $wG - e_0$ is the class of an effective divisor for at most finitely many elements wG of WG. This implies that $wG \cdot e_0 = \alpha(I(Z_{wG}))$ and hence that $\alpha(I(Z_{wG})) = \beta(I(Z_{wG})) = wG \cdot e_0$, for all but finitely many elements wG of WG. Thus all but finitely many of $\{Z_{wG} | w \in W\}$ are Cremona equivalent to Z; i.e., up to finite sets, the Cremona equivalence class of Z is $\{Z_{wG} | w \in W\}$, so Corollary III.5 follows from Theorem III.3. $\diamondsuit$

We end this section applying Corollary II.7 to uniform fat point subschemes. In particular, in Corollary III.8 and Corollary III.9 we obtain some evidence for Conjecture I.iii.2, based on the following version of Corollary II.7, for uniform fat point ideals at 10 or more expectedly good points.

Corollary III.6: *Let $p_1, \ldots, p_r$ be $r \geq 10$ distinct expectedly good points of $\mathbf{P}^2$, let $e_0, \ldots, e_r$ be the corresponding exceptional configuration, and let $I = I(mp_1 + \cdots + mp_r)$ with $m > 0$ be a fat points ideal. Let F denote $\alpha(I)e_0 - me_1 - \cdots - me_r$ and define μ, l_1, h and q_1 as in Lemma II.6. Then we have:*

- *(a) $\alpha(I) = \beta(I)$ unless $h = 1$ (in which case μ clearly has maximal rank);*
- *(b) the maximal rank property for I holds if and only if μ has maximal rank;*
- *(c) μ has maximal rank if and only if $\max(0, 2h - \alpha(I) - 2) = \dim \ker \mu$;*
- *(d) $l_1 \leq \max(0, 2h - \alpha(I) - 2) \leq \dim \ker \mu \leq l_1 + q_1$; and*
- *(e) $\max(0, 2h - \alpha(I) - 2) = l_1 + q_1$ unless $l_1 = 0$ and $q_1 > 0$.*

Proof: (a) First we show that any uniform class $G = de_0 - m(e_1 + \cdots + e_r)$ with $m > 0$ which is the class of an effective divisor is numerically effective (in particular, F is numerically effective). Recall that on the blow up X of expectedly good points the only prime divisors of negative self-intersection are the exceptional curves (that is, the smooth rational curves with self-intersection -1, each of which thus meets $-K_X$ once). Now note that $d > 3m$; otherwise, $-mK_X = G + (3m - d)e_0$ is the class of an effective divisor with negative self-intersection meeting positively every prime divisor of negative self-intersection, which is absurd. But $d > 3m$ means that G is the class of an effective divisor meeting every prime divisor of negative self-intersection positively. Thus G is numerically effective. (We also note two similarly proved facts that we will need below: since $G - (e_0 - e_1) = (d - 3m - 1)e_0 + e_1 - mK_X$, if $G - (e_0 - e_1)$ is the class of an effective divisor, it too is numerically effective; and $-mK_X + (d - 3m - 1)e_0 + (e_0 - e_1) = G - e_1$ so $G - e_1$ meets every exceptional curve nonnegatively, hence $G - e_1$ is also numerically effective if it is effective.)

Thus F is the class of an effective and numerically effective divisor. If $h^0(X, F) > 1$, we must show that $|F|$ is free. More generally, let $\mathtt{D}$ be any effective and numerically effective divisor on X with $h^0(X, \mathcal{D}) > 1$. We will show that $|\mathtt{D}|$ is fixed component free. Consider a Zariski decomposition $D = H + N$, where the class of the free part of $|D|$ is H and the class of the fixed part $\mathtt{N}$ is N. Suppose $\mathtt{E}$ is an exceptional curve which occurs as a component of $\mathtt{N}$; then $E \cdot H = 0$ (else $h^0(X, \mathcal{D}) = h^0(X, \mathcal{H})$ is impossible by Riemann–Roch), so $0 \le D \cdot E = N \cdot E$. Suppose $E \cdot C > 0$ for some other component $\mathtt{C} \ne \mathtt{E}$ of $\mathtt{N}$. Either $\mathtt{C}$ is numerically effective or it is exceptional, but $h^1(X, \mathcal{C}) = 0$ either way, so we have an exact sequence $0 \to H^0(X, C) \to H^0(X, C + E) \to H^0(\mathtt{E}, \mathcal{O}_{\mathtt{E}}(C + E)) \to 0$ from which the contradiction $1 = h^0(X, N) \ge h^0(X, C + E) > 1$ follows. Thus $\mathtt{E}$ is disjoint from the other components of $\mathtt{N}$, and hence $0 \le D \cdot E = N \cdot E < 0$. This contradiction shows that no exceptional curve is a component of $\mathtt{N}$. Therefore, N is numerically effective. Thus $h^1(X, N) = 0$, so $1 = h^0(X, N) = 1 + (N^2 - K_X \cdot N)/2$, which implies $(N^2 - K_X \cdot N)/2 = 0$. But $1 + (H^2 - K_X \cdot H)/2 = h^0(X, H) = h^0(X, H + N) = 1 + (H^2 - K_X \cdot H)/2 + H \cdot N + (N^2 - K_X \cdot N)/2$, which implies $H \cdot N = 0$. If $H^2 > 0$, then by the index theorem the subgroup of $\mathrm{Cl}(X)$ perpendicular to H is negative definite; since $N^2 \ge 0$, we must have $N = 0$. Similarly, if $H^2 = 0$ but $N^2 > 0$, then $H \cdot N = 0$ implies $H = 0$. Thus $H^2 = 0$ implies $N^2 = 0$ and so also $N \cdot K_X = 0$. Moreover, since $1 < h^0(X, H)$, $H^2 = 0$ implies $-K_X \cdot H > 0$. Now, the points p_i are expectedly good, hence strongly nonspecial, so, as in the proof of Lemma III.1(a), wN is, for some $w \in W$, a nonnegative integer linear combination of the classes H_i, $0 \le i \le r$. Since $N^2 = -K_X \cdot N = 0$, the only possibility is that wN is a nonnegative multiple of H_9. If $N \ne 0$, then we get the contradiction: $0 = H \cdot N = wH \cdot H_9 \ge wH \cdot H_r = wH \cdot w(-K_X) = -H \cdot K_X > 0$.

(b) Clearly, for $t < \alpha$ we have $I_t = 0$, so $I_t \otimes R_1 \to I_{t+1}$ has maximal rank. But the regularity of I is at most $\alpha + 1$ since F is numerically effective and our points are expectedly good, so $I_t \otimes R_1 \to I_{t+1}$ has maximal rank for $t > \alpha$ by Lemma II.5. Thus I has the maximal rank property if and only if $\mu : I_\alpha \otimes R_1 \to I_{\alpha+1}$ has maximal rank.

(c) Since $p_1, \ldots, p_r$ are expectedly good and F is numerically effective, it follows that $h^1(X, F) = 0$ so Corollary II.7(a) implies the result.

(d) Corollary II.7(b) gives $\max(0, 2h - \alpha(I) - 2) \le \dim \ker \mu \le l_1 + q_1$.

If $h^0(X, F - (e_0 - e_1)) = 0$, then $l_1 \le \max(0, 2h - \alpha(I) - 2)$ is clear, so suppose $h^0(X, F - (e_0 - e_1)) > 0$. Thus $F - (e_0 - e_1)$ is the class of an effective divisor, hence it is numerically effective, so $h^1(X, F - (e_0 - e_1)) = 0$. As in the proof of Corollary II.7(c) we have $h = l_1 + \alpha(I) + 1 - m$, so $(2h - \alpha(I) - 2) - l_1 = l_1 + \alpha(I) - 2m$. But $\alpha(I)^2 - rm^2 = F^2 \ge 0$ implies $\alpha(I) - 2m > 0$, and we now see $2h - \alpha(I) - 2 > l_1$, which implies $l_1 \le \max(0, 2h - \alpha(I) - 2)$.

(e) If $q_1 = 0$, then (d) implies the result, so let $q_1 > 0$. If also $l_1 > 0$, then $F - e_1$ and $F - (e_0 - e_1)$ are classes of effective divisors, hence (as we saw above) numerically effective, so $h^1(X, F - e_1) = 0 = h^1(X, F - (e_0 - e_1))$, so the result follows by Corollary II.7(c). ◇

Remark III.7: Whereas the bound $\max(0, 2h - \alpha(I) - 2) \le \dim \ker \mu$ in Corollary III.6 is in fact exactly what one obtains from [Cam], the upper bound $\dim \ker \mu \le l_1 + q_1$ is always at least as good as Campanella's (which is always either $h - 1$ or $h - 2$), and except in extremal cases (i.e., $h \le 2$ or $\alpha \le h \le \alpha + 1$) it is better.

Assuming expectedly good points, computer runs suggest that $\max(0, 2h - \alpha(I) - 2)$ equals $l_1 + q_1$ fairly often, possibly for infinitely many m for each $r > 9$ which is not an even square. The next two corollaries verify this possibility for some special values of r.

Corollary III.8: *Using the notation and hypotheses of Corollary III.6, μ has maximal rank for infinitely many m whenever $r + i$ is an odd square for some $i \in \{-3, -2, -1, 0, 1, 2, 3, 4\}$.*

Proof: First assume $r+i$ is an odd square for some $i \in \{0,1,2,3,4\}$; then it is not hard to see that there is an odd integer $2t+1 \in [\sqrt{r}, \sqrt{r}+2/\sqrt{r})$. By Corollary III.6, μ has maximal rank whenever $q_1 = 0$, so for the given r it suffices to check that $q_1 = 0$ for infinitely many m.

By the proof of Corollary III.6, $F-e_1$ is numerically effective whenever it is effective. Taking cohomology of $0 \to \mathcal{O}_X(F-e_1) \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_{\mathtt{E}_1} \to 0$ we see the restriction map $H^0(X,\mathcal{F}) \to H^0(E_1, \mathcal{F} \otimes \mathcal{O}_{\mathtt{E}_1})$ always has maximal rank. Thus $h^0(X,\mathcal{F}) \le h^0(\mathtt{E}_1, \mathcal{F} \otimes \mathcal{O}_{\mathtt{E}_1})$ implies that $q_1 = 0$.

To apply this, note that $h^0(\mathtt{E}_1, \mathcal{F} \otimes \mathcal{O}_{\mathtt{E}_1}) = m+1$ and, since F is numerically effective by the proof of Corollary III.6, that $h^0(X,\mathcal{F}) = \binom{\alpha(I)+2}{2} - r\binom{m+1}{2}$. From this we obtain the following criterion: $q_1 = 0$ for each m for which $0 < \binom{x+2}{2} - r\binom{m+1}{2} \le m+1$ has a positive integer solution x.

Now, we know $b^2 - rm^2 = \epsilon$ has infinitely many positive integer solutions (b,m), where we take $\epsilon = 0$ if r is a square and we take $\epsilon = 1$ otherwise (in which case we have Pell's equation). Substituting $b+t-1$ in our criterion for x and simplifying gives $-t^2-t-\epsilon < (2t+1)b - rm \le 2m+2-t^2-t-\epsilon$.

Since $b \ge \sqrt{r}m$ and $2t+1 \ge \sqrt{r}$, we clearly have $-t^2-t-\epsilon < (2t+1)b - rm$. Since $b < \sqrt{r}m+1$, we see $(2t+1)b - rm < (2t+1)(\sqrt{r}m+1) - rm$; i.e., $(2t+1)b - rm$ is bounded above by a linear function of m. Using $2t+1 < \sqrt{r}+2/\sqrt{r}$ shows that the coefficient of m in this linear function is less than 2, so for m sufficiently large we have $(2t+1)b - rm < 2m+2-t^2-t-\epsilon$.

Now assume $r+i$ is an odd square for some $i \in \{-1,-2,-3\}$; then it is not hard to see that there is an odd integer $2t-1 \in (\sqrt{r}-2/\sqrt{r}, \sqrt{r})$. By Corollary III.6 it suffices to check that $l_1 > 0$ for infinitely many m, so this time we use the fact that $F-(e_0-e_1)$ is numerically effective whenever it is effective. From the proof of Corollary III.6(d), we have $l_1 = \max(0, h-\alpha+m-1)$. Thus $h-\alpha+m-1 > 0$ implies $l_1 > 0$, which gives us the following criterion: $l_1 > 0$ for each m for which $x < \binom{x+1}{2} - r\binom{m+1}{2} + m \le x+m$ has a positive integer solution x. (If x is a solution, then $x = \alpha+1$. In particular, the second inequality fails for $x > \alpha+1$, while the first fails for $x < \alpha$.) Simplifying gives $x+(r-2)m < x^2 - rm^2 \le x+rm$, and as above, $b^2 - rm^2 = 1$ has infinitely many positive integer solutions (b,m). Substituting $b+t$ in for x and simplifying gives $2m-(t-t^2-1) > rm-(2t-1)b \ge -t+t^2+1$.

Since $b \ge \sqrt{r}m$, we have $rm-(2t-1)b \le rm-(2t-1)\sqrt{r}m$, so $rm-(2t-1)b$ is bounded above by a linear function of m, and using $2t-1 > \sqrt{r}-2/\sqrt{r}$ shows that the coefficient of m in this linear function is less than two. It now follows that our criterion's first inequality holds for all sufficiently large m. For the other inequality, using $b < \sqrt{r}m+1$ shows $rm-(2t-1)b$ is strictly bounded below by a linear function of m, and now using $2t-1 < \sqrt{r}$ shows the coefficient of m in this linear function is positive. Thus the second inequality also holds for all sufficiently large m. ♢

Corollary III.8 gives a partial answer to the question of for which r do our bounds infinitely often force μ to have maximal rank. An interesting side remark here is that in fact the bounds force μ to have maximal rank for all but finitely many m when r is an odd square, whereas for r an even square our bounds never force maximal rank. A slightly different approach (and further easy variations of it) gives additional examples, such as $r = 13$.

Corollary III.9: *Using the notation and hypotheses of Corollary III.6, μ has maximal rank for infinitely many m whenever $r = (ca)^2 + 4c^2 > 9$ for positive odd integers a and c.*

Proof: Here we use the criterion developed in the proof of Corollary III.8 involving $q_1 = 0$. So substitute $x = \sqrt{r}m + t - 1$ into $0 < \binom{x+2}{2} - r\binom{m+1}{2} \le m+1$. For t real, this has solutions for all sufficiently large integers $m > 0$ if t is in the interval $[(\sqrt{r}-1)/2, (\sqrt{r}-1)/2 + 1/\sqrt{r})$, and so for m sufficiently large $0 < \binom{x+2}{2} - r\binom{m+1}{2} < m+1$ has a positive integer solution x if the interval $[\sqrt{r}m + (\sqrt{r}-1)/2, \sqrt{r}m + (\sqrt{r}-1)/2 + 1/\sqrt{r})$ contains an integer. After simplifying, this is equivalent to finding an integer λ such that

$$0 \le \frac{2\lambda+1}{2m+1} - \sqrt{r} < \frac{2}{\sqrt{r}(2m+1)}.$$

It is well known that $\sqrt{r}$ has infinitely many rational approximations p/q accurate to order $1/q^2$ if r is not a square. The problem here is to ensure in addition that p and q are odd with $p/q > \sqrt{r}$. Whether this also is known we do not know, but it can at least be verified in certain cases. For example, consider the continued fraction expansion

$$\sqrt{(ca)^2 + 4c^2} = ca + \cfrac{2c}{a + \cfrac{1}{a + \cdots}}.$$

Taking successive convergents (see [Bk] for background on continued fractions) gives a sequence $\{c_i\}$ of rational approximations which for $i \equiv 2 (\mathrm{mod}\ 6)$ is a ratio $p/q > \sqrt{r}$ of odd integers p and q. Moreover, the general theory of continued fractions implies that each convergent p/q is accurate to order $1/q^2$. Thus for $r = (ca)^2 + 4c^2 > 9$ expectedly good points, μ has maximal rank for infinitely many m. $\diamondsuit$

IV. Results for $r < 10$

Finally, we prove complete results for arbitrary symbolic powers of ideals of $r \leq 9$ general points of $\mathbf{P}^2$; i.e., for ideals $I(Z)$ of fat point subschemes $Z = mp_1 + \cdots + mp_r$ for $r \leq 9$ general points p_i. Along the way we prove Theorem I.iii.1 and we conclude by proving Corollary I.1.

We divide our analysis into three cases, $r \leq 5$, $6 \leq r \leq 8$, and $r = 9$, with the second case requiring most of the effort but also being the most interesting.

IV.i. Five or Fewer General Points

Let X be the blow up of $\mathbf{P}^2$ at $r \leq 5$ general points. By Lemma II.2 and Remark II.3, we can compute $\mathcal{S}(\mathcal{F}, e_0)$ for an arbitrary class F if we can do so whenever F is a numerically effective class. But any five or fewer general points in the plane lie on a smooth conic, so the results of [Cat] apply. Translating the results of [Cat] to the language used here and examining what [Cat] proves, we find that $\mathcal{S}(\mathcal{F}, e_0) = 0$ for any numerically effective class F. (In fact, [Cat] iteratively finds generators for and a resolution of $I(Z)$ for any fat point subscheme $Z = m_1p_1 + \cdots + m_tp_t$, where $p_1, \ldots, p_t$ are distinct points of a smooth plane conic, which includes the case of a uniform Z supported at 5 or fewer general points of $\mathbf{P}^2$. From our perspective, the key fact in [Cat], not explicitly stated there, is that $\mathcal{S}(\mathcal{F}, e_0) = 0$ for any numerically effective class F on the blow up X of points on a smooth conic. See [H7] for an explicit proof and a generalization.)

Applying the foregoing to $Z = m(p_1 + \cdots + p_r)$ for $r \leq 5$ general points $p_1, \ldots, p_r \in \mathbf{P}^2$ and $m > 0$, we have the following. Since $F = \beta(I(Z))e_0 - m(e_1 + \cdots + e_r)$ is numerically effective and hence $\mathcal{S}(\mathcal{F}, e_0) = 0$, we see μ_β is surjective and so has maximal rank; thus the RUMRP holds for $r \leq 5$. As for the UMRP, for $r = 1$ it is easy to see that $I(Z)_t = 0$ for $t < m$ and that $te_0 - me_1$ is numerically effective for $t \geq m$. The former means that μ_t is injective for $t < m$, and by the preceding paragraph and numerical effectivity of $te_0 - me_1$ for $t \geq m$, we have $\mathcal{S}(te_0 - me_1, e_0) = 0$ for $t \geq m$, and hence μ_t is surjective for $t \geq m$. Thus the UMRP holds on $\mathbf{P}^2$ for $r = 1$. For $r = 4$, $I(Z)_t = 0$ for $t < 2m$, since $2e_0 - (e_1 + \cdots + e_4)$ is numerically effective but $[2e_0 - (e_1 + \cdots + e_4)] \cdot [te_0 - m(e_1 + \cdots + e_4)] < 0$. Also, $\mathcal{S}(\mathcal{F}_t, e_0) = 0$ for $F_t = te_0 - m(e_1 + \cdots + e_4)$ with $t \geq 2m$, since $F_t = m(2e_0 - (e_1 + \cdots + e_4)) + (t - 2m)e_0$ is numerically effective. Thus the UMRP holds on $\mathbf{P}^2$ also for $r = 4$.

To see that the UMRP on $\mathbf{P}^2$ fails for $r = 2, 3, 5$, it is enough by Proposition II.4 to find in each case a uniform abnormal class. But these have already been exhibited in Remark II.3: for $r = 2$, we have $e_0 - (e_1 + e_2)$; for $r = 3$, there is $3e_0 - 2(e_1 + e_2 + e_3)$; and for $r = 5$, $2e_0 - (e_1 + \cdots + e_5)$. One can check, in fact, that for $r = 2, 3, 5$, $I(m(p_1 + \cdots + p_r))$ fails to have the maximal rank property if and only if: $r = 2$ and $m \geq 2$; or $r = 3$ or $r = 5$ and $m \geq 3$.

IV.ii. Six to Eight General Points

Theorem IV.ii.1 determines $\mathcal{S}(\mathcal{F}, e_0)$ for any numerically effective uniform class on a blow up X of $\mathbf{P}^2$ at $6 \leq r \leq 8$ general points. That the RUMRP holds for $r = 6$ but not for $r = 7$ or 8 follows directly from Theorem IV.ii.1. That the UMRP fails for $6 \leq r \leq 8$ follows from Proposition II.4, since, as shown in Remark II.3, in each case the blow up of r general points supports a uniform abnormal class: for $r = 6$,

$E = 12e_0 - 5(e_1 + \cdots + e_6)$ is such; for $r = 7$, $E = 21e_0 - 8(e_1 + \cdots + e_7)$ is such; and for $r = 8$, $E = 48e_0 - 17(e_1 + \cdots + e_8)$ is such.

Theorem IV.ii.1: *Let $F = F(d, m, r)$ be a uniform numerically effective class on the blowing up X of $6 \leq r \leq 8$ general points of $\mathbf{P}^2$ (where $F(d, m, r)$ denotes $de_0 - m(e_1 + \cdots + e_r)$).*

- *(a) If $r = 6$, then $\mathcal{R}(e_0, \mathcal{F})\mathcal{S}(e_0, \mathcal{F}) = 0$.*
- *(b) If $r = 7$, then $\mathcal{R}(e_0, \mathcal{F})\mathcal{S}(e_0, \mathcal{F}) = 0$ unless $F = lF(8, 3, 7)$ for $l \geq 3$, in which case $\mathcal{S}(e_0, \mathcal{F}) = 7$.*
- *(c) If $r = 8$, then $\mathcal{R}(e_0, \mathcal{F})\mathcal{S}(e_0, \mathcal{F}) = 0$, unless $F = lF(17, 6, 8)$ for $l \geq 9$, in which case $\mathcal{S}(e_0, \mathcal{F}) = 48$, or unless $F = lF(17, 6, 8) + F(3, 1, 8)$ for $l \geq 6$, in which case $\mathcal{S}(e_0, \mathcal{F}) = 16$.*

Proof: Note that we can compute h^0 for any class F on X, as discussed in Remark II.3 or more generally using [H1], keeping in mind that any 8 or fewer general points are expectedly good [H5].

So let $6 \leq r \leq 8$ and let $F = de_0 - m(e_1 + \cdots + e_r)$ be a uniform class. If F is numerically effective, then $h^1(X, F + te_0) = 0$ for all $t \geq 0$ (by Remark II.3), so $\mathcal{S}(F + te_0, e_0) = 0$ for all $t > 0$ by Lemma II.5. Thus we only need to consider $\delta e_0 - m(e_1 + \cdots + e_r)$, where δ is the least d such that $de_0 - m(e_1 + \cdots + e_r)$ is numerically effective. We will denote this class as F_m, or just by F if our meaning is clear. Using Remark II.3 it follows that δ is the least positive integer d such that: $d \geq 5m/2$ if $r = 6$; $d \geq 8m/3$ if $r = 7$; or $d \geq 17m/6$ if $r = 8$.

First say $r = 6$ and $\mathtt{E}$ is the effective divisor whose class is E, where here we take $E = 12e_0 - 5(e_1 + \cdots + e_6)$. Note that $\mathtt{E}$ is a disjoint union of six (-1)-curves. Also, if m is odd, then $F_m = -K_X + (m-1)(5e_0 - 2(e_1 + \cdots + e_6))/2$, while $F_m = m(5e_0 - 2(e_1 + \cdots + e_6))/2$ if m is even. In any case, $h^2(X, F_m - e_0) = 0$ by duality.

If m is odd, one checks (by induction on m in $0 \to \mathcal{O}_X(F_m - e_0) \to \mathcal{O}_X(F_{m+2} - e_0) \to \mathcal{O}_X(\mathcal{F}_{m+2} - e_0) \otimes \mathcal{O}_{\mathtt{C}}) \to 0$, where the class of the smooth rational curve $\mathtt{C}$ is F_2) that, suppressing the subscript, $h^1(X, F - e_0) = 0$, and hence (by Lemma II.5) that $\mathcal{S}(\mathcal{F}, e_0) = 0$ so suppose $m = 2s$, with $s \geq 1$. For $s = 2$, $e_0 \cdot (F - E + e_0) = -1$, so $h^0(X, F - E + e_0) = 0$ so $\mathcal{S}(F - E, e_0) = 0$. For $s > 2$, $F - E$ is numerically effective with odd uniform multiplicity, so $\mathcal{S}(F - E, e_0) = 0$ by the preceding case. Since $\mathcal{F} \otimes \mathcal{O}_{\mathtt{E}} = \mathcal{O}_{\mathtt{E}}$, it is easy to check that $\mathcal{S}(\mathcal{O}_{\mathtt{E}}, e_0) = 0$, using Proposition II.1(b) applied to the components of $\mathtt{E}$. If we now check that $h^1(X, F - E + e_0) = 0$ and $h^1(X, F - E) = 0$, then we can apply Proposition II.1(a) to $(0 \to \Gamma(F - E) \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F} \otimes \mathcal{O}_{\mathtt{E}}) \to 0) \otimes \Gamma(e_0)$ to obtain $\mathcal{S}(\mathcal{F}, e_0) = 0$. But for $s > 2$, we have $h^1(X, F - E + e_0) = 0$ and $h^1(X, F - E) = 0$ by Remark II.3. For $s = 2$, we have $F - E = K_X + e_0$ and $F - E + e_0 = K_X + 2e_0$; now using duality and descending to $\mathbf{P}^2$ we see $h^1(X, F - E + ae_0) = h^1(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-a - 1)) = 0$ for any a.

We are left with the case $s = 1$, thus $m = 2$, but here $(F - (e_0 - e_1)) \cdot F < 0$ and $(F - e_1) \cdot F < 0$ so $l_1 = q_1 = 0$ and, by Corollary II.7, $\mathcal{R}(\mathcal{F}, e_0) = 0$.

Now say $r = 7$ and $\mathtt{E}$ is the effective divisor whose class is $E = 21e_0 - 8(e_1 + \cdots + e_7)$. This time $\mathtt{E}$ is a union of seven disjoint (-1)-curves and $F_m = s\mathcal{F}_3 - tK_X$, where $F_3 = 8e_0 - 3(e_1 + \cdots + e_7)$ and the integers s and t are defined by taking $m = 3s + t$ such that $0 \leq t < 3$.

For $t = 2$ and any $s \geq 0$ we have $h^1(X, F_m - e_0) = 0$ (as in the case $r = 6$), which gives $\mathcal{S}(\mathcal{F}_m, e_0) = 0$ by Lemma II.5. For $m < 9$, we have $\mathcal{R}(\mathcal{F}_m, e_0)\mathcal{S}(\mathcal{F}_m, e_0) = 0$ (with, in fact, $\mathcal{S}(\mathcal{F}_m, e_0) = 0$ when m is not 3 or 6) by computing cohomology and applying Corollary II.7. Similarly, for $m = 10$ we have $\mathcal{S}(\mathcal{F}_m, e_0) = 0$, so applying Proposition II.1(a) with $m = 10$ to $(0 \to \Gamma(F_m - E) \to \Gamma(\mathcal{F}_m) \to \Gamma(\mathcal{F}_m \otimes \mathcal{O}_{\mathtt{E}}) \to 0) \otimes \Gamma(e_0)$ shows that $\mathcal{S}(\mathcal{F}_m \otimes \mathcal{O}_{\mathtt{E}}, e_0) = 0$. But $F_{3s+1} \otimes \mathcal{O}_{\mathtt{E}} = F_{10} \otimes \mathcal{O}_{\mathtt{E}}$ for any s, hence $\mathcal{S}(\mathcal{F}_{3s+1} \otimes \mathcal{O}_{\mathtt{E}}, e_0) = 0$ for any s. Checking $\mathcal{S}(\mathcal{F}_{3s} \otimes \mathcal{O}_{\mathtt{E}}, e_0) = 7$ is even easier, using $\mathcal{F}_{3s} \otimes \mathcal{O}_{\mathtt{E}} = \mathcal{O}_{\mathtt{E}}$. We can now handle the remaining cases, $0 \leq t \leq 1$ with $s \geq 3$; for these we consider $(0 \to \Gamma(F_m - E) \to \Gamma(\mathcal{F}_m) \to \Gamma(\mathcal{F}_m \otimes \mathcal{O}_{\mathtt{E}}) \to 0) \otimes \Gamma(e_0)$, using $m = 3s + t$, $F_m - E = F_{3(s-3)+t+1}$ and $\mathcal{F}_m \otimes \mathcal{O}_{\mathtt{E}} = \mathcal{O}_X(-tK_X) \otimes \mathcal{O}_{\mathtt{E}}$. By induction, $\mathcal{S}(\mathcal{F}_{3(s-3)+t+1}, e_0) = 0$, so by the exact sequence of Proposition II.1(a) we obtain $\mathcal{S}(\mathcal{F}_{3s+1}, e_0) = 0$ and $\mathcal{S}(\mathcal{F}_{3s}, e_0) = 7$.

In conclusion, for $r = 7$, $\mathcal{R}(\mathcal{F}, e_0)\mathcal{S}(\mathcal{F}, e_0) = 0$ for all numerically effective uniform classes F except $F = lF_3$ for $l \geq 3$, in which case $\mathcal{S}(\mathcal{F}, e_0) = 7$.

We now proceed to the last case, for which $X \to \mathbf{P}^2$ is a blow up of $r = 8$ general points of $\mathbf{P}^2$. Here we let $\mathtt{E}$ be the effective divisor whose class is $E = 48e_0 - 17(e_1 + \cdots + e_8)$; $\mathtt{E}$ is a union of eight disjoint (-1)-curves, each of which under $X \to \mathbf{P}^2$ maps to a plane sextic with seven double points and a triple point. Here we have $F_m = sF_6 - tK_X$, where $F_6 = 17e_0 - 6(e_1 + \cdots + e_8)$ and $m = 6s + t$ with $0 \leq t < 6$. It follows for $s \geq 3$ that we have $F_{6s+t} - E = F_{6(s-3)+t+1}$.

We first need to compute $\mathcal{S}(\mathcal{F}_m \otimes \mathcal{O}_{\mathtt{E}}, e_0)$ and $\mathcal{R}(\mathcal{F}_m \otimes \mathcal{O}_{\mathtt{E}}, e_0)$ for $0 \leq t < 5$. Let C be the class of any component $\mathtt{C}$ among the eight components of $\mathtt{E}$. Then $\mathcal{S}(\mathcal{F}_m \otimes \mathcal{O}_{\mathtt{E}}, e_0) = 8\mathcal{S}(\mathcal{F}_m \otimes \mathcal{O}_{\mathtt{C}}, e_0)$, so we restrict our attention to $\mathtt{C}$. Note that $\mathcal{F}_m \otimes \mathcal{O}_{\mathtt{C}} = \mathcal{O}_X(-tK_X) \otimes \mathcal{O}_{\mathtt{C}} = \mathcal{O}_{\mathtt{C}}(t)$.

For $t = 0$, clearly $\mathscr{R}(\mathcal{O}_{\mathtt{C}}, e_0) = 0$ (a linear form times a nonzero constant cannot vanish on a sextic) whence $\mathscr{S}(\mathcal{O}_{\mathtt{C}}, e_0) = 4$, so consider $t = 1$. Then we have $\mathscr{R}(\mathcal{O}_{\mathtt{C}}(1), e_0) = 0$ and so $\mathscr{S}(\mathcal{O}_{\mathtt{C}}(1), e_0) = 2$: letting x and y be a basis for $\Gamma(\mathcal{O}_{\mathtt{C}}(1))$, a nontrivial element of $\Gamma(\mathcal{O}_{\mathtt{C}}(1)) \otimes \Gamma(e_0)$ which maps to 0 in $\Gamma(\mathcal{O}_{\mathtt{C}}(1) \otimes e_0) = \Gamma(\mathcal{O}_{\mathtt{C}}(7))$ gives an equation $xf = yg$, where f and g are restrictions to C of distinct lines in $\mathbf{P}^2$. But f and g have degree 6, so $xf = yg$ implies f and g have 5 zeros on $\mathtt{C}$ in common. Since the image of $\mathtt{C}$ in $\mathbf{P}^2$ has at most a triple point, two distinct lines can have at most 3 points of $\mathtt{C}$ in common, contradicting there being a nontrivial element of the kernel.

For $t = 2$, both $\mathcal{R}(\mathcal{O}_{\mathtt{C}}(2), e_0)$ and $\mathcal{S}(\mathcal{O}_{\mathtt{C}}(2), e_0)$ vanish: let x and y be as before and let f, g, h be a basis for the restriction of $\Gamma(e_0)$ to $\mathtt{C}$ such that f and g correspond to lines in $\mathbf{P}^2$ which meet at the triple point of the image of $\mathtt{C}$ in $\mathbf{P}^2$. If $\mathscr{R}(\mathcal{O}_{\mathtt{C}}(2), e_0) \neq 0$, then we have an equation $q_1 f + q_2 g + q_3 h = 0$, where q_1, q_2, q_3 (not all 0) lie in the span of $\{x^2, xy, y^2\}$. Since f and g have exactly 3 zeros in common, we cannot have $q_3 = 0$, and so h also has a zero in common with f and g, which gives the contradiction that the restriction of $\Gamma(e_0)$ to $\mathtt{C}$ has a base point. Thus $\mathscr{R}(\mathcal{O}_{\mathtt{C}}(2), e_0) = 0$ from which we easily compute $\mathscr{S}(\mathcal{O}_{\mathtt{C}}(2), e_0) = 0$.

For $t = 3, 4$ or 5, we have $\mathscr{S}(\mathcal{O}_{\mathtt{C}}(t), e_0) = 0$: say $t = 3$ ($t = 4$ or 5 are similar). Let x and y be as above; thus cubics in x and y span $\Gamma(\mathcal{O}_{\mathtt{C}}(3))$. But $\Gamma(\mathcal{O}_{\mathtt{C}}(1)) \otimes \Gamma(\mathcal{O}_{\mathtt{C}}(2))$ surjects onto $\Gamma(\mathcal{O}_{\mathtt{C}}(3))$, and, by the previous case, $\Gamma(\mathcal{O}_{\mathtt{C}}(2)) \otimes \Gamma(e_0)$ surjects onto $\Gamma(\mathcal{O}_{\mathtt{C}}(8))$, so $\Gamma(\mathcal{O}_{\mathtt{C}}(3)) \otimes \Gamma(e_0)$ and $(\Gamma(\mathcal{O}_{\mathtt{C}}(1)) \otimes \Gamma(\mathcal{O}_{\mathtt{C}}(2))) \otimes \Gamma(e_0)$ and $\Gamma(\mathcal{O}_{\mathtt{C}}(1)) \otimes \Gamma(\mathcal{O}_{\mathtt{C}}(8))$ all have the same image in $\Gamma(\mathcal{O}_{\mathtt{C}}(9))$. Since $\mathtt{C}$ is rational, we know $\Gamma(\mathcal{O}_{\mathtt{C}}(1)) \otimes \Gamma(\mathcal{O}_{\mathtt{C}}(8))$ surjects onto $\Gamma(\mathcal{O}_{\mathtt{C}}(9))$, whence $\mathscr{S}(\mathcal{O}_{\mathtt{C}}(t), e_0) = 0$.

Now we are ready to consider $\mathscr{S}(\mathcal{F}_m, e_0)$. First, $h^1(X, F_{6s+5} - e_0) = 0$ for $s \geq 0$, hence $\mathcal{S}(\mathcal{F}_{6s+5}, e_0) = 0$ for all $s \geq 0$ by Lemma II.5. For $m = 2, 8$, or 14 we apply Proposition II.1 to $(0 \to \Gamma(F_m - C) \to \Gamma(\mathcal{F}_m) \to \Gamma(\mathcal{F}_m \otimes \mathcal{O}_{\mathtt{C}}) \to 0) \otimes \Gamma(e_0)$, where we take C to be, respectively, $6e_0 - 3e_1 - 2e_2 - \cdots - 2e_8$, $24e_0 - 9(e_1 + \cdots + e_4) - 8(e_5 + \cdots + e_8)$ and $42e_0 - 15(e_1 + \cdots + e_7) - 14e_8$, from which it follows that $\mathscr{S}(\mathcal{F}_m, e_0)$ is, respectively, 0, 1 and 0, from which we derive that $\mathscr{R}(\mathcal{F}_m, e_0)\mathscr{S}(\mathcal{F}_m, e_0) = 0$. We also check $\mathscr{R}(\mathcal{F}_m, e_0)\mathscr{S}(\mathcal{F}_m, e_0) = 0$ for $21 \leq m \leq 22$ and for the remaining values of $0 \leq m \leq 18$ by applying Corollary II.7 (we suppress the explicit computations).

It turns out, in fact, that $\mathcal{S}(\mathcal{F}_{6s+t}, e_0) = 0$ for $3 \leq t \leq 4$ and $0 \leq s \leq 3$. Thus, using $\mathcal{S}(\mathcal{F}_{6s+5}, e_0) = 0$ and $\mathcal{S}(\mathcal{F}_{6s+t} \otimes \mathcal{O}_E, e_0) = 0$ for $3 \leq t \leq 4$ from above, with $(0 \to \Gamma(F_m - E) \to \Gamma(\mathcal{F}_m) \to \Gamma(\mathcal{F}_m \otimes \mathcal{O}_{\mathtt{E}}) \to 0) \otimes \Gamma(e_0)$ and Proposition II.1 and induction, we conclude $\mathcal{S}(\mathcal{F}_{6s+t}, e_0) = 0$ for $3 \leq t \leq 4$ and all $s \geq 0$. In particular, we now see that either $\mathcal{R}(\mathcal{F}_m \otimes \mathcal{O}_{\mathtt{E}}, e_0) = 0$ or $\mathcal{S}(F_m - E, e_0) = 0$ for every $m \geq 18$, and hence that $0 \to \mathcal{S}(\mathcal{F}_{6(s-3)+t+1}, e_0) \to \mathcal{S}(\mathcal{F}_{6s+t}, e_0) \to \mathcal{S}(\mathcal{F}_{6s+t} \otimes \mathcal{E}, e_0) \to 0$ is exact for all $0 \leq t < 5$ and any $s \geq 3$.

For $s \geq 3$, we thus obtain three recursion formulas: $\mathscr{S}(\mathcal{F}_{6s+2}, e_0) = \mathscr{S}(\mathcal{F}_{6(s-3)+3}, e_0) + \mathscr{S}(\mathcal{F}_{6s+2} \otimes \mathcal{O}_{\mathtt{E}}, e_0)$ (or $\mathscr{S}(\mathcal{F}_{6s+2}, e_0) = 0$, since $\mathscr{S}(\mathcal{F}_{6(s-3)+3}, e_0)$ and $\mathscr{S}(\mathcal{F}_{6s+2} \otimes \mathcal{O}_{\mathtt{E}}, e_0)$ vanish); $\mathscr{S}(\mathcal{F}_{6s+1}, e_0) = \mathscr{S}(\mathcal{F}_{6(s-3)+2}, e_0) +$ 16 (since $\mathscr{S}(\mathcal{F}_{6s+1} \otimes \mathcal{O}_{\mathtt{E}}, e_0) = 16$); and $\mathscr{S}(\mathcal{F}_{6s}, e_0) = \mathscr{S}(\mathcal{F}_{6(s-3)+1}, e_0) + 32$ (since $\mathscr{S}(\mathcal{F}_{6s} \otimes \mathcal{O}_{\mathtt{E}}, e_0) = 32$). Thus we see that $\mathscr{S}(\mathcal{F}_{6s+2}, e_0) = 0$ for $s > 1$, that $\mathscr{S}(\mathcal{F}_{6s+1}, e_0) = 16$ for $s > 4$, and that $\mathscr{S}(\mathcal{F}_{6s}, e_0) = 48$ for $s > 7$. It is also now easy to check that $\mathscr{R}(\mathcal{F}_{6s+1}, e_0) = 0$ if and only if $s < 6$, and that $\mathscr{R}(\mathcal{F}_{6s}, e_0) = 0$ if and only if $s < 9$. Thus with our results above we have that $\mathscr{R}(\mathcal{F}_m, e_0)\mathscr{S}(\mathcal{F}_m, e_0) = 0$ for every $m \geq 0$ except when $m = 6s$ and $s \geq 9$, in which case $\mathscr{S}(\mathcal{F}_m, e_0) = 48$, or when $m = 6s + 1$ and $s \geq 6$, in which case $\mathscr{S}(\mathcal{F}_m, e_0) = 16$. ◊

Example IV.ii.2: Here we use our results to explicitly compute a resolution's modules. Consider eight general points, each taken with multiplicity $m = 205$; thus $Z = 205(p_1 + \cdots + p_8)$. Then $I(Z)$ has: $\nu_{579} = 10$ generators in degree 579 (since 579 is the first degree d such that $I(Z)_d \neq 0$, and we have $\dim_k I(Z)_{579} = 10$); $\nu_{580} = 201$ (since $579e_0 - 205(e_1 + \cdots + e_8)$ has free part $H = 51e_0 - 18(e_1 + \cdots + e_8)$ and fixed part $N = 528e_0 - 187(e_1 + \cdots + e_8)$, and here $\mathscr{S}(e_0, H + N) = \mathscr{S}(e_0, \mathcal{H}) + (h^0(X, e_0 + H + N) - h^0(X, H + e_0)) = 33 + 168$); $\nu_{581} = 208$ (since, for $d = 580$, $H = 340e_0 - 120(e_1 + \cdots + e_8)$ and $N = 240e_0 - 85(e_1 + \cdots + e_8)$, and $\mathscr{S}(e_0, H + N) = 48 + 160$); and $\nu_{582} = 16$ (since, for $d = 581$, $H = 581e_0 - 205(e_1 + \cdots + e_8)$ and $N = 0$, and $\mathscr{S}(e_0, H + N) = 16 + 0$). Moreover, the regularity of $I(Z)$ is 582, so there are no other generators. (These numbers can be compared with Campanella's bounds [Cam]: $10 \leq \nu_{579} \leq 10$, $201 \leq \nu_{580} \leq 210$, $70 \leq \nu_{581} \leq 280$, and $0 \leq \nu_{582} \leq 79$.)

Now, a minimal free resolution of $I(Z)$ has the form $0 \to F_1 \to F_0 \to I(Z) \to 0$, and our data show that $F_0 = R^{10}[-579] \oplus R^{201}[-580] \oplus R^{208}[-581] \oplus R^{16}[-582]$. We therefore now know the Hilbert function of F_0, which with the Hilbert function of $I(Z)$ determines the Hilbert function of F_1, from which we recover F_1 itself as $R^{138}[-581] \oplus R^{216}[-582] \oplus R^{80}[-583]$.

IV.iii. Nine General Points

Nine general points of $\mathbf{P}^2$ always lie on a smooth cubic curve, so let X be obtained by blowing up nine distinct points $p_1, \ldots, p_9$ of a smooth cubic curve in $\mathbf{P}^2$. As indicated in Remark II.3, $-K_X$ is the class of a smooth elliptic curve $\mathtt{C}$, and any uniform class $F = de_0 - m(e_1 + \cdots + e_9)$ on X with $m > 0$ can be written as $F = te_0 - mK_X$ (with $t = d - 3m$). As in Remark II.3, if the restriction of $\mathcal{O}_X(-tK_X)$ to C has infinite order in Pic($\mathtt{C}$), define a to be 0. If the order l is finite, define a by requiring $m = al + b$ with $0 \leq b < l$.

Theorem IV.iii.1: *Let $F = te_0 - mK_X$ with $m > 0$, where X is the blowing up of 9 distinct points of a smooth cubic in $\mathbf{P}^2$, with a defined as above.*

(a) *If $t < -1$, then $\mathcal{S}(\mathcal{F}, e_0) = 0$.*
(b) *If $t = -1$, then $\mathcal{s}(\mathcal{F}, e_0) = a + 1$ and $\mathcal{R}(\mathcal{F}, e_0) = 0$.*
(c) *If $t = 0$, then $\mathcal{s}(\mathcal{F}, e_0) = 3m - 3a$ and $\mathcal{R}(\mathcal{F}, e_0) = 0$.*
(d) *If $t > 0$, then $\mathcal{s}(\mathcal{F}, e_0) = 0$.*

Proof: Part (a) follows since then $h^0(X, F + e_0) = 0$, part (b) follows since then $\mathcal{s}(\mathcal{F}, e_0) = h^0(X, F + e_0) = a + 1$, while parts (c) and (d) follow from Theorem 3.2.1.2 of [H7], and, in the case of (c), from Lemma II.2 and Remark II.3. ♢

Corollary IV.iii.2: *The UMRP (and hence the RUMRP) holds on $\mathbf{P}^2$ for $r = 9$.*

Proof: By Theorem IV.iii.1, we see $\mathcal{R}(\mathcal{F}, e_0)\mathcal{s}(\mathcal{F}, e_0) = 0$ for any uniform class on the blowing up X of any 9 distinct points of a smooth cubic. Thus $I(Z)$ has the maximal rank property for any uniform fat point subscheme Z supported at nine general points of $\mathbf{P}^2$. ♢

IV.iv. Proof of Corollary I.1

We close with the proof of Corollary I.1. Let $p_1, \ldots, p_r$ be $r \leq 9$ general points of $\mathbf{P}^2$. Then Theorem IV.ii.1 shows that $I(m(p_1 + \cdots + p_r))$ fails to have the maximal rank property if: $r = 7$, $m = 3l$ and $3 \leq l \leq 7$; or $r = 8$, $m = 6l$ and $9 \leq l \leq 16$; or $r = 8$, $m = 6l + 1$ and $6 \leq l \leq 13$. Using Remark II.3 we check in each of these cases that $\alpha(I) = \beta(I)$.

Conversely, since $r \leq 9$ and $\alpha(I) = \beta(I)$, I has the maximal rank property if and only if $\mu_{\beta(I)}$ has maximal rank. By Subsection IV.i, Theorem IV.ii.1 and Corollary IV.iii.2, if $\mu_{\beta(I)}$ fails to have maximal rank, then either $r = 7$, $m = 3l$ and $3 \leq l$, or $r = 8$, $m = 6l$ and $9 \leq l$, or $r = 8$, $m = 6l + 1$ and $6 \leq l$. Using Remark II.3 to restrict to those cases with $\alpha(I) = \beta(I)$ gives the result.

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