THE HILBERT FUNCTION OF A REDUCED $k$-ALGEBRA

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Introduction

Let $A = \bigoplus_{i \geq 0} A_i$ be a graded $k$-algebra of finite type (where $k$ is a field, $A_0 = k$ and $A$ is generated as a $k$-algebra by $A_1$). Such algebras have been called standard $G$-algebras by Stanley [7]. They are those $k$-algebras which can be written in the form $A = k[X_0, \ldots, X_n]/I$, where the $X_i$ are indeterminates of degree 1 and $I$ is a homogeneous ideal. The Hilbert function $\{b_i\}, i \geq 0$, of $A$ is defined by $b_i = \dim_k A_i$. Macaulay [4] characterized the Hilbert function of a standard $G$-algebra. In this paper we characterize the Hilbert function of a reduced standard $G$-algebra.

Our proof is based on properties of points in ‘generic position’ on an algebraic variety and on a partition of a differentiable 0-sequence into two differentiable pieces. These techniques are of some interest in their own right.

A preliminary version of some of the results in this paper can be found in the preprints [5, 6].

1. Review of 0-sequences

Let $h$ and $i$ be positive integers. Then $h$ can be written uniquely in the form

(1.1) \[ h = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j} \]

where $m_i > m_{i-1} > \ldots > m_j \geq j \geq 1$ [3]. This expression for $h$ is called the $i$-binomial expansion of $h$.

Also, define

(1.2) \[ h^{(i)} = \binom{m_i+1}{i+1} + \binom{m_{i-1}+1}{i} + \cdots + \binom{m_j+1}{j+1} \]

and $0^{(i)} = 0$.

Definition 1.3. A sequence of non-negative integers $\{c_i\}, i \geq 0$, is called an 0-sequence if $c_0 = 1$ and $c_{i+1} \leq c_i^{(i)}$ for all $i \geq 1$.

Macaulay's theorem (as expressed in modern form by Stanley [7, Theorem 2.2]) is as follows.

Theorem 1.4. The following are equivalent:

(a) $\{c_i\}, i \geq 0$, is an 0-sequence;
(b) $\{c_i\}, i \geq 0$, is the Hilbert function of a standard $G$-algebra.

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A useful way to think about the function \( h^{(\alpha)} \) is as follows. First write down Pascal's triangle

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & \cdots \\
1 & 2 & 3 & 4 & 5 & \cdots \\
1 & 3 & 6 & 10 & 15 & \cdots \\
1 & 4 & 10 & 20 & 35 & \cdots \\
1 & 5 & 15 & 35 & 70 & \cdots \\
\vdots & & & & & \\
\end{array}
\]

(best written in rectangular form).

The rows and columns are both numbered beginning with 0. The entry in the \( i \)-th row and \( j \)-th column is \( \binom{i+j}{j} \). The term \( \binom{m}{i} \) in the \( i \)-binomial expansion of \( h \) is the largest entry not exceeding \( h \) in the \( i \)-th column. Write \( h = \binom{m}{i} + r \) with \( r \geq 0 \). If \( r = 0 \) we are done. Otherwise, repeat the process with \( r \) and column \( i - 1 \). Ultimately the \( i \)-binomial expansion of \( h \) is achieved. The terms in the \( i \)-binomial expansion of \( h \) occur in consecutive columns \( i, i-1, \ldots, j \), where \( i \geq j \geq 1 \). If \( l \) is chosen so that \( j \leq l < i \) and the \( l, l+1 \) terms lie, respectively, in rows \( r_l, r_{l+1} \), then \( r_l \leq r_{l+1} \). Shifting each term to the right and adding yields \( h^{(\alpha)} \).

For example, the bold numbers in the above array are the terms of the 3-binomial expansion of 18 and so \( 18^{(3)} = 15 + 10 + 3 = 28 \). Using the description above it is easy to verify the following properties of \( h^{(\alpha)} \).

**Theorem 1.6.** (a) Let \( \alpha, \beta, i \) be positive integers with \( \alpha < \beta \). Then \( \alpha^{(\beta)} < \beta^{(\alpha)} \).

(b) Let \( \{b_i\}, i \geq 0 \), be an 0-sequence with \( b_1 = m \). Then

\[
b_{i+1} \leq \binom{m+i}{i+1} \quad \text{for every } i \geq 0.
\]

(c) Let \( h = \binom{(n-1)+i}{i} - 1 \). Then

\[
h^{(\alpha)} = \binom{n+i}{i+1} - n.
\]

Now suppose that \( A \) is a reduced standard \( G \)-algebra, with \( k \) infinite, and Hilbert function \( \{b_i\}, i \geq 0 \). Then \( A \) contains a non-zerodivisor \( x \) of degree 1 and \( A/xA \) has Hilbert function \( \{1, b_1-1, b_2-b_1, \ldots\} \). This motivates the following.

**Definition 1.7.** Let \( \{b_i\}, i \geq 0 \), be an 0-sequence. Then \( \{b_i\} \) is **differentiable** if the difference sequence \( \{c_i\}, c_i = b_i - b_{i-1} \), is again an 0-sequence. (We adopt the convention that \( b_{-1} = 0 \).)
Thus, a necessary condition that \( \{b_i\} \) be the Hilbert function of a reduced standard \( G \)-algebra (for \( k \) infinite) is that \( \{b_i\} \) be differentiable. We shall prove in §3 that this necessary condition is also sufficient.

For simplicity in the exposition we introduce the following definition.

**Definition 1.8.** The 0-sequence \( \{b_i\}, i \geq 0 \), has dimension \( d \) if there is a polynomial \( f(x) \), with rational coefficients, of degree \( d \) with the property that for all \( n \gg 0 \), \( f(n) = b_n \).

The fact that every 0-sequence has a dimension is an immediate consequence of Macaulay's theorem (1.4 above) and a theorem of Hilbert (see for example [8, p. 232, Theorem 41]).

2. Points in generic position on an algebraic variety

Let \( I \) be a homogeneous ideal in \( k[x_0, \ldots, x_n] = R \) and let \( A = R/I \). Then \( X = \text{Proj}(A) = V(I) \) is a closed subscheme of \( \mathbb{P}^n_k \). The Hilbert function of \( A \) will sometimes be written \( \{H_A(i)\} \) or \( \{H_x(i)\} \). The map \( I \to V(I) \) gives a one to one correspondence between homogeneous radical ideals in \( R \) and reduced closed subschemes of \( \mathbb{P}^n_k \). Reduced closed subschemes of \( \mathbb{P}^n_k \) will be referred to as algebraic varieties. If \( X \) is an algebraic variety, the corresponding ideal will be denoted by \( I(X) \). A closed subset of \( \mathbb{P}^n_k \) will be given its reduced subscheme structure unless stated otherwise.

Let \( I \) and \( J \) be two homogeneous ideals of \( R \). Then we have an exact sequence

\[
0 \to R/I \cap J \to (R/I) \oplus (R/J) \to R/(I + J) \to 0.
\]

For \( r, s \in R \), the maps are \( \bar{r} \to (\bar{r}, \bar{r}) \) and \( (\bar{r}, \bar{s}) \to \bar{r} - \bar{s} \), the bars denoting the class in the appropriate quotient ring. Suppose that \( R/I \cap J, R/I, R/J \) and \( R/(I + J) \) have Hilbert functions \( \{b_i\}, \{c_i\}, \{d_i\} \) and \( \{e_i\} \) respectively. Using the fact that if \( e_s = 0 \) then \( b_i = 0 \) for \( i \geq s \), we obtain the following.

**Lemma 2.2.** With the above notation,

\[
c_i + d_i = b_i + e_i \quad \text{for} \quad i \geq 0.
\]

If \( e_s = 0 \) then \( b_i = c_i + d_i \) for \( i \geq s \).

The assertion that \( e_s = 0 \) for some \( s \) is equivalent to \( V(I) \cap V(J) = \emptyset \) which, in turn, is equivalent to \( \sqrt{I + J} = (x_0, \ldots, x_n) \).

**Lemma 2.3.** Let \( V \subset X \) be two subvarieties of \( \mathbb{P}^n_k \).

(a) \( H_V(i) \leq H_X(i) \) for \( i \geq 0 \).

(b) Let \( P \) be a \( k \)-rational point not in \( V \). There is an integer \( d \) such that

\[
H_{V \cup \{P\}}(i) = H_V(i) \quad \text{for} \quad i < d
\]

and

\[
H_{V \cup \{P\}}(i) = H_V(i) + 1 \quad \text{for} \quad i \geq d.
\]
(c) Let \( k \) be algebraically closed and let \( e \) be the least integer such that \( H_V(e) < H_X(e) \). There exists a \( k \)-rational point \( P \in X - V \) for which \( d = e \) (with \( d \) as in (b) above).

**Proof.** Let \( I = I(V) \) and \( L = I(X) \). Then (a) holds because there is a surjection \( R/L \to R/I \). Let \( J = I(P) \). Then (b) is immediate from 2.2 once we observe that \( R/J \cong k[t] \), \( R/(I + J) \cong k[t]/(t^d) \) for some \( d > 0 \) and \( I(V \cup \{P\}) = I \cap J \). To prove (c), let \( F \) be a (non-zero) form of degree \( e \) in the kernel of \( R/L \to R/I \) and choose \( P \in X \) such that \( F(P) \neq 0 \).

**Definition 2.4.** The \( s \) \( k \)-rational points \( P_i \in X \) \((1 \leq i \leq s)\) are in generic position on \( X \) if the Hilbert function of \( V = \bigcup_{i=1}^{s} P_i \) satisfies

\[
H_V(i) = \inf \{ H_X(i), s \} \quad \text{for } i \geq 0.
\]

**Theorem 2.5.** There exist \( s \) \( k \)-rational points in generic position on \( X \) in each of the following cases:

(a) \( k \) is algebraically closed and \( X \) contains more than \( s \) points;

(b) \( k \) is infinite and \( X = \mathbb{P}^n_k \), \( n \geq 1 \);

(c) \( k \) is arbitrary and \( X \) is a union of \( t \geq s \) \( k \)-rational points.

**Proof.** We prove (a) by choosing \( s \) \( k \)-rational points one after the other as in Lemma 2.3(c). For (c) above note that Lemma 2.3(c) still holds because every point in \( X \) is \( k \)-rational. Generic position in \( \mathbb{P}^n_k \) coincides with the concept of generic \( s \)-position introduced in [2], where (b) is proved.

**Examples 2.6.** (a) Let \( X = \mathbb{P}^1_k \) or let \( X \) be a non-singular conic in \( \mathbb{P}^2_k \). Then every finite set of \( k \)-rational points in \( X \) is in generic position on \( X \).

(b) Let \( X \) be a rational non-singular quartic in \( \mathbb{P}^3_k \) (not contained in a hyperplane). A hyperplane intersects \( X \) in four points, which are not in generic position on \( X \).

Note that the curves \( X \) in (a), (b) are isomorphic as abstract varieties. Thus the notion of "generic position on \( X \)" depends on the projective embedding of \( X \) (as was to be expected).

**Theorem 2.7.** Let \( X \) be a subvariety of \( \mathbb{P}^n_k \) and let \( H \) be a hyperplane of \( \mathbb{P}^n_k \) not containing any irreducible component of \( X \). Let \( V \) be a subvariety of \( H \) such that

\[
H_V(i) = \binom{i+n-1}{n-1} \quad \text{for } i \leq s
\]

(that is, generic in \( H \) for \( i \leq s \)). Then

\[
H_{X \cup V}(i) = H_V(i) + H_X(i-1) \quad \text{for } i \leq s.
\]
Proof. Let \( J = I(X), J' = I(X \cup V) \) and let \( L \) be a form of degree 1 defining \( H \). Suppose that \( G \) is a form of degree \( i \leq s \) vanishing on \( X \cup V \). By the generic assumption on \( V \), \( G \) vanishes on \( H \) and so contains \( L \) as a factor. Write \( G = LF \). Since \( H \) does not contain any irreducible component of \( X \), we must have \( F \in J_{i-1} \). That is, \( J'_i = (LJ)_i = LJ_{i-1} \). Theorem 2.7 now follows from the degree \( i \) portion of the exact sequence

\[
0 \longrightarrow L/LJ \longrightarrow R/LJ \longrightarrow R/L \longrightarrow 0
\]

and the fact that \((L/LJ)_i \cong (R/J)_{i-1}\).

**Corollary 2.8.** If we maintain the hypotheses and notation of 2.7 and assume, moreover, that \( H_X(s-1) = H_X(s) \) (that is if \( k \) is algebraically closed then \( X \) consists of \( H_X(s-1) \) points), then

\[
H_{X \cup V}(i) = H_V(i) + H_X(i-1) \quad \text{for } i \leq s,
\]

and

\[
H_{X \cup V}(i) = H_V(i) + H_X(i) \quad \text{for } i \geq s.
\]

Proof. The first conclusion is Theorem 2.7, while the second follows from 2.2 once we observe that

\[
H_{X \cup V}(s) = H_V(s) + H_X(s-1) = H_V(s) + H_X(s).
\]

### 3. The Hilbert function of a reduced \( k \)-algebra

Let \( S = \{b_i\} \) be a differentiable 0-sequence, with \( b_1 = n + 1 \geq 2 \). We construct, from \( S \), two new sequences \( S_1 \) and \( S'_1 \).

First let \( d_i = \binom{n+i-1}{i}, i \geq 0 \) (that is the Hilbert function of a polynomial ring in \( n \) variables) and set \( c_i = b_{i+1} - d_{i+1} \). There are two possibilities.

**Case 1,** in which \( c_i \leq c_{i+1} \) for all \( i \geq 0 \). In this case let \( S_1 = \{c_i\}, i \geq 0 \).

**Case 2,** in which \( 1 = c_0 \leq c_1 \leq c_2 \leq \ldots \leq c_{k-1} \) and \( c_{k-1} > c_k \). In this case let \( S_1 \) be the sequence \( c_0, c_1, \ldots, c_{k-1} \rightarrow \) (where the arrow indicates that, if \( i \geq h-1 \), then the \( i \)-th term of \( S_1 \) is \( c_{h-1} \)).

We can maintain a uniform notation if in Case 1 we set \( h = \infty \) and regard sets like \( \{i \mid i \geq \infty\} \) to be empty.

With this convention in mind we now define \( S'_1 = \{c'_i\}, i \geq 0 \), where

\[
c'_i = \begin{cases} 
  d_i & \text{if } i \leq h, \\
  b_i - c_{h-1} & \text{if } i \geq h.
\end{cases}
\]

(The two definitions agree if \( i = h \).)
Lemma 3.1. Let $S = \{b_i\}, i \geq 0$, be a differentiable 0-sequence with $b_1 = n + 1$ and let $h$ be as in the above construction of $S_1$. Let $\{a_i\}$ be the difference sequence of $S$ (that is $a_i = b_i - b_{i-1}$ for $i \geq 0$, $b_{-1} = 0$). Then

(a) \( \binom{r+n-1}{r+1} \leq a_{r+1} \leq \binom{r+n}{r+1} \) if $1 \leq r \leq h-1$;
(b) \( a_{h+1} < \binom{n+h-1}{h+1} \);
(c) \( a_{(r+1)+1} = \left[ a_{r+1} - \binom{r+n+1}{r+1} \right]^{(r)} + \binom{n+r}{r+2}, 1 \leq r \leq h-1. \)

Proof. The right inequality of (a) is true for all $r$ by 1.6(b). For all $r$ we have $b_{r+1} = c_r + d_{r+1}$ and so $a_{r+1} = c_r + \binom{n+r-1}{r+1}$ (where $\{c_i\}$ is the difference sequence of $\{c_i\}$). Part (b) and the remaining inequality of (a) now follow from the assumptions that $c_r \geq 0$ for $r \leq h-1$ and $c_{h-1} > c_h$ (that is $c_h < 0$).

Now (a) implies either that $a_{r+1} = \binom{r+n}{r+1}$ in which case (c) can be checked directly, or that the first term in the $(r+1)$-binomial expansion of $a_{r+1}$ is $\binom{r+n-1}{r+1}$. In the latter case (c) follows immediately from the definition of $a_{(r+1)+1}$.

Theorem 3.2. Let $S = \{b_i\}, i \geq 0$, with $b_1 = n + 1$, be a differentiable 0-sequence. Then the sequences $S_1$ and $S_1'$ constructed above are both differentiable 0-sequences.

Proof. First consider $S_1$. Let $\{\beta_i\}$ be the difference sequence of $S_1$. It suffices to show that $\{\beta_i\}$ is an 0-sequence. Also, since $\beta_r = 0$ for $r \geq h$ it suffices to show that $\beta_{r+1} \leq \beta^{(r)}_{r}$ for $1 \leq r \leq h-2$. Then $\beta^{(r)}_{r} = (\tilde{c}_r)^{(r)}$ (in the notation of 3.1) and

\[ (\tilde{c}_r)^{(r)} = \left( a_{r+1} - \binom{n+r-1}{r+1} \right)^{(r)} = a_{(r+1)+1} - \binom{n+r}{r+2} \]

by 3.1(c). But, since $\{a_i\}$ is an 0-sequence, we have

\[ a_{(r+1)+1} - \binom{n+r}{r+2} \leq a_{r+2} - \binom{n+r}{r+2} = \beta_{r+1} \]

and so $\beta_{r+1} \leq \beta^{(r)}_{r}$.

Now consider $S_1'$. Again it suffices to show that the first difference of $S_1'$ is an 0-sequence and the only non-trivial verification there is that $c_{h+1} - c_h \leq (c_h - c_{h-1})^{(h)}$. Writing this out one sees that this follows from 3.1(b).

Theorem 3.3. Let $k$ be an infinite field and let $S = \{b_i\}, i \geq 0$, be a differentiable 0-sequence with $b_1 = n + 1$. There is a radical ideal $I$ in $k[x_0, \ldots, x_n]$ such that $S$ is the Hilbert function of $k[x_0, \ldots, x_n]/I$. 
**Proof.** We prove the theorem by induction. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. To any differentiable 0-sequence we associate an element $(a, b) \in \mathbb{N} \times \mathbb{N}$ where $a \geq 2$ and $b \geq 1$. We do this as follows.

To the sequence $T = \{t_i\}$, $t_i = \left(\begin{array}{c} t_i + i - 1 \\ i \end{array}\right)$, $i \geq 0$, we associate the element $(t_i + 1, 1)$. Such an 0-sequence is the Hilbert function of a polynomial ring in $t_i$ variables. We call such an 0-sequence generic.

If $T = \{t_i\}$, $i \geq 0$, is a non-generic differentiable 0-sequence, we associate to $T$ the element $(a, b) = (a(T), b(T))$, where $a = t_1$ and $b$ is the smallest integer such that $t_h \leq b$. The theorem is true for $S$ associated to $(a, 1)$, $a \geq 1$ (such $S$ are generic) and for $S$ associated to $(2, l)$, $l \geq 2$ (such an $S$ is the Hilbert function of $l$ points on a line).

Now, let $S = \{b_i\}$, $i \geq 0$, be a non-generic differentiable 0-sequence and let $S_1$ and $S'_1$ be the differentiable 0-sequences constructed above. It is clear from the construction that $a(S'_1) = a(S) - 1$ and either $a(S_1) < a(S)$ or $a(S_1) = a(S)$ and $b(S_1) = b(S) - 1$.

Thus $(a(S'_1), b(S'_1)) < (a(S), b(S))$ and $(a(S_1), b(S_1)) < (a(S), b(S))$ (in the usual lexicographic ordering). Thus, by induction, we may find a subvariety $X$ of $\mathbb{P}^{n-1}$ with Hilbert function $S_1$. Let $H$ be a hyperplane not containing any irreducible component of $X$ (since $k$ is infinite, such an $H$ exists) and let $V$ be a subvariety of $H$ with Hilbert function $S'_1$. By Corollary 2.8, $X \cup V$ has Hilbert function $S$; this completes the proof.

If we carry out the induction all the way to the $(2, l)$, $l \geq 2$, and $(a, 1)$, $a \geq 1$, sequences, we obtain the following.

**Corollary 3.4.** Let $k$ be an infinite field and let $S = \{b_i\}$, $i \geq 0$, be a $d$-dimensional differentiable 0-sequence with $b_1 = n + 1$. There is a $d$-dimensional variety $V \subset \mathbb{P}^n_k$, which is the union of a finite number of linear subvarieties of dimension \( \leq d \), having Hilbert function $S$.

**Example 3.5.** Let $S$ be the zero-dimensional differentiable 0-sequence

$$S : 1 \ 3 \ 6 \ 10 \ 14 \ 17 \ 20 \ 21 \ 22 \rightarrow .$$

The procedure we have described gives $S$ as the Hilbert function of 22 points in $\mathbb{P}^2$ where 9 are on a line $L_1$, 6 are on a line $L_2$, 5 are on a line $L_3$ and 2 are on a line $L_4$ where the $L_i$ are any four distinct lines of $\mathbb{P}^2$ and no point chosen on $L_i$ lies on $L_j$ for $j < i$.

**Example 3.6.** If $S$ is the 1-dimensional differentiable 0-sequence

$$S : 1 \ 4 \ 10 \ 19 \ 30 \ 40 \ 50 \rightarrow$$

(continuing with increments of 10) then our procedure gives $S$ as the Hilbert function of the union of 10 lines lying in a hyperplane of $\mathbb{P}^3$, 20 points on another hyperplane and 5 points on a third hyperplane.
This $S$ can also be shown to be the Hilbert function of the 10 lines of a double-five configuration on a non-singular cubic surface in $\mathbb{P}^3$. (See [1] for a description of the double-five.)

4. The conductor of a finite set of points in $\mathbb{P}^r$

Let $X = \{P_1, \ldots, P_s\}$ be any set of $s$ $k$-rational points in $\mathbb{P}_k^r$, with homogeneous coordinate ring $A$. The integral closure of $A$ is $\bar{A} = \prod_{i=1}^s k[t_i]$ and the conductor of $A$ in $\bar{A}$ is of the form $\prod_{i=1}^s t_i^nk[t_i]$, for some integers $n_i \geq 1$. We shall refer to $n_i$ as the degree of the conductor of $P_i$ in $X$. Let $V = X - P_i$. It is clear that

$$H_V(i) = \begin{cases} H_X(i) & \text{for } 0 \leq i < n_i, \\ H_X(i) - 1 & \text{for } i \geq n_i. \end{cases}$$

Thus $n_i$ is the 'd' of Lemma 2.3(b).

This motivates the following definition.

**Definition 4.1.** Let $S = \{b_i\}, i \geq 0$, be a zero-dimensional differentiable 0-sequence. We say that $l$ is a permissible value for $S$ if the sequence $\tilde{S} = \{c_i\}$, where $c_i = b_i$ for $0 \leq i < l$ and $c_i = b_i - 1$ for $i \geq l$, is again a differentiable 0-sequence.

Hence, the possible $n_i$ are a subset of the permissible values for the Hilbert function of $A$.

**Example 4.2.** Let $S$ be the zero-dimensional differentiable 0-sequence $S : 1367 \rightarrow$.

The permissible values are 2 and 3. Seven points of $\mathbb{P}^2$ in uniform position [2] have Hilbert function $S$ and each point has degree of conductor 3.

Thus, not every permissible value need occur as the degree of conductor for some point. However, we do have the following.

**Theorem 4.3.** Let $S$ be a zero-dimensional differentiable 0-sequence and let $k$ be an infinite field. If $X$ is a finite set of $k$-rational points with Hilbert function $S$ and constructed by Corollary 3.4 then every permissible value for $S$ is the degree of the conductor for some $P \in X$.

**Proof.** If $S$ is $1 \ 2 \ 3 \ldots \ n \rightarrow$, then the only permissible value is $n-1$, which must be the degree of conductor of any point in $X$.

Otherwise, let $S_1$ and $S'_1$ be the sequences constructed from $S$ in §3 and let $l$ be a permissible value for $S$. It is straightforward to prove that either $l$ is a permissible value for $S'_1$ or $l-1$ is a permissible value for $S_1$. We have $X = X_1 \cup X'_1$, where $X_1$ and $X'_1$ are constructed by applying the procedures of §3 (see Corollary 3.4). By induction on the number of points we conclude that either there is a point $P \in X'_1$
with degree of conductor \( l - 1 \) in \( X_1 \) or there is a point \( P \in X'_1 \) with degree of conductor \( l \) in \( X'_1 \). By Corollary 2.8 and Lemma 2.3(b) the point \( P \) has degree of conductor \( l \) in \( X \).

5. Some geometry associated to the Hilbert function of a set of points

Let \( S = \{b_i\}, i \geq 0 \), be a \( d \)-dimensional differentiable 0-sequence. The variety \( X \) with Hilbert function \( S \) that is produced by Corollary 3.4 is obviously special if \( d > 0 \). Proposition 5.1 below suggests that \( X \) is, in some sense, extremal even if \( d = 0 \).

**Proposition 5.1.** Let \( S = \{b_i\}, i \geq 0 \), be a zero-dimensional differentiable 0-sequence with \( b_1 = n + 1 \) and \( b_m = s \) for \( m \gg 0 \). Let \( d \) be the least integer for which \( b_d = s \).

(i) If \( Y = \{P_1, \ldots, P_s\} \) is any set of points in \( \mathbb{P}^n \) with Hilbert function \( S \) then at most \( d + 1 \) of the points of \( Y \) can be collinear.

(ii) If \( X = \{Q_1, \ldots, Q_s\} \) is a set of points in \( \mathbb{P}^n \) having Hilbert function \( S \) and constructed as in Corollary 3.4, then exactly \( d + 1 \) points of \( X \) are collinear.

**Proof.** Lemma 2.3(b) easily gives (i), and (ii) is obvious if \( n = 1 \). Hence to prove (ii) we may assume that \( n \geq 2 \).

Let \( S' = \{c_i\} \) be produced from \( S \) as in §3. Then \( c_i < b_1 \) and \( d \) is also the degree at which \( S' \) becomes constant. The proof now follows by induction on \( n \).

In general there need not be \( d + 1 \) collinear points. However, for some \( S \), the \( X \) produced by Corollary 3.4 is the only one possible, as we now show.

**Proposition 5.2.** Let \( S = \{b_i\}, i \geq 0 \), \( b_1 = n + 1 \), be a zero-dimensional differentiable 0-sequence for which \( b_{d-2} = s - 2 \), \( b_{d-1} = s - 1 \) and \( b_i = s \) for \( i \geq d \).

(i) If \( X = \{P_1, \ldots, P_s\} \) is any set of \( s \) points in \( \mathbb{P}^n \) having Hilbert function \( S \) then \( X \) contains a subset \( Y \) of \( d + 1 \) collinear points.

(ii) If, moreover, \( n = 2 \) then \( X \setminus Y \) has Hilbert function \( \{c_i\} \) where

\[
 c_i = \begin{cases} 
 b_{i+1} - (i + 2) & \text{for } 0 \leq i \leq d, \\
 s - (d + 1) & \text{for } i > d.
\end{cases}
\]

**Proof.** Note that (ii) follows from (i) by 2.8, and so it suffices to prove (i).

From Theorem 2.5(c) there are two points \( P_1, P_s \) in \( X \) such that \( X \setminus \{P_1, P_s\} \) has Hilbert function

\[
 1 \ b_1 \ b_2 \ \ldots \ b_{d-2} = s - 2 \rightarrow.
\]

Thus, \( X \setminus \{P_i\}, i = 1, s \), has Hilbert function

\[
 1 \ b_1 \ b_2 \ \ldots \ b_{d-2} = s - 2 \ b_{d-1} = s - 1 \rightarrow.
\]

Let \( L \) be the line joining \( P_1 \) and \( P_s \). The proof is by induction on the number of points of \( X \) that we have found on \( L \) (starting with \( i = 2, P_1 \) and \( P_s \)).
Suppose we have found \( i \) points \( \{P_1, P_2, ..., P_{i-1}, P_s\} \) on \( L \) \((2 \leq i < d + 1)\). The points \( P_1, ..., P_{i-1} \) are collinear, and hence have Hilbert function
\[
1 \quad 2 \quad 3 \quad ... \quad (i-1) \rightarrow .
\]

Now \( \{P_1, P_2, ..., P_{i-1}\} \subset X \setminus \{P_s\} \) and \( i-2 \leq d - 2 \), and so by Lemma 2.3(c) we can adjoin \( s - i - 1 \) new points of \( X \setminus \{P_s\} \) to \( \{P_1, P_2, ..., P_{i-1}\} \) to obtain \( R = \{P_1, ..., P_{i-1}, R_1, ..., R_{s-i-1}\} \), a set of \( s - 2 \) points whose Hilbert function at \( d - 2 \) has value exactly \( s - 2 \). Let \( Q \) be the remaining point of \( X \setminus \{P_s\} \) not in \( R \). Let \( F \) be a form of degree \( d - 2 \) vanishing at all points of \( R \) except \( P_s \) and let \( G \) be any linear form vanishing at \( Q \) and \( P_s \). Then \( FG \) is a form of degree \( d - 1 \) vanishing at all points of \( X \) except possibly \( P_s \). But \( X \setminus \{P_s\} \) and \( X \) have the same Hilbert function in degree \( d - 1 \); hence \( FG(P_s) = 0 \) and so \( G(P_s) = 0 \). Thus \( Q \in L \), and this completes the induction and the proof.

**Example 5.3.** Consider the zero-dimensional differentiable 0-sequence
\[
S: 1 \quad 3 \quad 6 \quad 9 \quad 11 \quad 13 \quad 14 \quad 15 \rightarrow.
\]
The procedures of §3 construct a set \( X \) in \( \mathbb{P}^2 \) having Hilbert function \( S \) and consisting of 8 points on one line, 5 points on another line and 2 points on a third line. Proposition 5.2 tells us that any set of 15 points in \( \mathbb{P}^2 \) with Hilbert function \( S \) must be so situated.

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